On totally geodesic unit vector fields.

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Let T_1M be a unit tangent bundle of M endowed with Sasaki metric [8]. If ξ is a unit vector field on M, then one may consider ξ as a mapping $\xi : M \to T_1M$. The image $\xi(M)$ is a submanifold transverse to fibers in T_1M with metric induced from T_1M. Conversely, a manifold transverse to fibers in the (unit) tangent bundle can be given as image of some (unit) vector field on the base manifold [1]. Thus, a transverse to fibers submanifold in T_1M^n always can be locally represented by a unit vector field.

A unit vector field ξ is said to be *minimal* if $\xi(M)$ is a minimal submanifold in T_1M. A unit vector field on S^3 tangent to fibers of Hopf fibration $S^3 \stackrel{S^1}{\to} S^2$ is a unique one with globally minimal volume [4]. This result fails in higher dimensions. A lower volume has a vector field with one singular point. This field is a stereographic projection inverse image of parallel vector field on E^n [7]. The lowest volume has the *North-South* vector field with two singular points [3]. In [10] the author found the second fundamental form of $\xi(M)$ and presented some examples of vector fields with constant mean curvature. This result is a key to solve a problem on *totally geodesic unit vector fields* on a given Riemannian manifold. In [11] this question was treated in a case of 2-manifolds of constant curvature and in [13] was found an example of totally geodesic unit vector field on a surface of revolution with non-constant but sign-preserving Gaussian curvature.

In this note we drive the differential equation in covariant derivatives on a unit vector field such that its solution provides a totally geodesic property for $\xi(M^n)$

Let ξ be a fixed *unit* vector field on Riemannian manifold M^n. Denote by $A_\xi : T_qM^n \to \xi_q^\perp$ a point-wise linear operator, acting as

$$A_\xi X = \quad \nabla_X \xi$$

In case of integrable distribution $\xi^\perp$, the operator A_ξ is symmetric and is known as Wiengarten or a shape operator for each hypersurface of the foliation.

In general, A_ξ is not symmetric, but formally preserves the Codazzi equation. Namely, a covariant derivative of A_ξ is defined by

$$(\nabla_X A_\xi)Y = \nabla_X \nabla_Y \xi \quad \nabla_{\nabla_X Y}\xi. \tag{1}$$

Then for the curvature operator of M^n we can write down the non-holonomic Codazzi equation

$$R(X,Y)\xi = (\nabla_Y A_\xi)X \quad (\nabla_X A_\xi)Y.$$

Remark, that the right hand side is, up to constant, a *skew symmetric part* of covariant derivative of A_ξ.

Introduce a symmetric tensor field

$$Hess_\xi(X,Y) = \frac{1}{2}\big[(\nabla_Y A_\xi)X + (\nabla_X A_\xi)Y\big], \tag{2}$$

which is a *symmetric part* of covariant derivative of A_ξ. The trace

$$\sum_{i=1}^{n} Hess_\xi(e_i,e_i) := \Delta\xi,$$

where $e_1,\dots e_n$ is an orthonormal frame, is known as *rough Laplacian* [2] of the field ξ. Therefore, one can treat the tensor field (2) as a *rough Hessian* of the field. A vector field is called *harmonic*, if it is a critical point of energy functional of mapping $\xi : M^n \to T_1M^n$. Up to an additive constant, this functional is a total bending of a unit vector field [9] and the unit vector field is harmonic if and only if $\Delta\xi = \ \ |\nabla\xi|^2\xi$, where $|\nabla\xi|^2 = \sum_{i=1}^n |\nabla_{e_i}\xi|^2$ with respect to orthonormal frame $e_1,\dots e_n$ [9].

Introduce a tensor field

$$Hm_\xi(X,Y) = \frac{1}{2}\big[R(\xi,\nabla_X\xi)Y + R(\xi,\nabla_Y\xi)X\big], \tag{3}$$

which is a symmetric part of tensor field $R(\xi,\nabla_X\xi)Y$. The trace

$$\Delta H_\xi := \sum_{i=1}^{n} Hm_\xi(e_i,e_i)$$

is responsible for harmonicity of mapping $\xi : M^n \to T_1M^n$. Precisely, a harmonic unit vector field ξ defines a harmonic map $\xi : M^n \to T_1M^n$ if and only if $\Delta H_\xi = 0$ [5]. From this viewpoint, it is natural to call the tensor field (3) as *harmonicity tensor* of the field ξ.

Definition 1 *A unit vector field ξ on Riemannian manifold M^n is called totally geodesic if the image of (local) imbedding $\xi : M^n \to T_1M^n$ is totally geodesic submanifold in the unit tangent bundle T_1M^n with Sasaki metric.*

Now we can state a basic condition under which a given unit vector field ξ generates a totally geodesic submanifold in T_1M^n.

Proposition 1 *Let M^n be Riemannian manifold and T_1M^n its unit tangent bundle with Sasaki metric. Let ξ a smooth (local) unit vector field on M^n. The vector field ξ generates a totally geodesic submanifold $\xi(M^n) \subset T_1M^n$ if and only if ξ satisfies*

$$Hess_\xi(X,Y) = A_\xi Hm_\xi(X,Y) + \big\langle A_\xi X, A_\xi Y\big\rangle\, \xi$$

for all (local) vector fields X and Y on M^n.

Proof. The differential of mapping $\xi : M^n \to TM^n$ is acting as

$$\xi_* X = X^h + (\nabla_X \xi)^v = X^h \quad (A_\xi X)^v, \tag{4}$$

where ∇ means Levi-Civita connection on M^n and the lifts are considered to points of $\xi(M^n)$. It is well known that if ξ is a unit vector field on M^n, then the vertical lift ξ^v is a *unit normal* vector field on a hypersurface $T_1M^n \subset TM^n$. Since ξ is of unit length, $\xi_* X \perp \xi^v$ and hence, in fact, $\xi_* : TM^n \to T(T_1M^n)$.

Denote by $A^t_\xi : \xi_q^\perp \to T_qM^n$ a formal adjoint operator

$$\big\langle A_\xi X, Y\big\rangle = \big\langle X, A^t_\xi Y\big\rangle.$$

Then for each $Z \in \xi_q^\perp$ the vector field

$$\tilde Z = (A^t_\xi Z)^h + Z^v$$

is normal to $\xi(M^n)$.

Evidently, $\xi(M^n)$ is totally geodesic in T_1M^n if and only if at each point $q \in M^n$

$$\big\langle\big\langle \tilde\nabla_{\xi_* X}\, \xi_* Y, \tilde Z\big\rangle\big\rangle = 0$$

where $\tilde\nabla$ is the Levi-Civita connection of Sasaki metric on TM^n. To calculate $\tilde\nabla_{\xi_* X}\, \xi_* Y$, use formulas [6], namely,

$$\begin{array}{ll} \tilde\nabla_{X^h} Y^h = (\nabla_X Y)^h \quad \frac{1}{2}(R(X,Y)\xi)^v, & \tilde\nabla_{X^v} Y^h = \frac{1}{2}(R(\xi,X)Y)^h, \\ \tilde\nabla_{X^h} Y^v = (\nabla_X Y)^v \ + \frac{1}{2}(R(\xi_1,Y)X)^h, & \tilde\nabla_{X^v} Y^v = 0. \end{array} \tag{5}$$

A direct calculation yields

$$\tilde\nabla_{\xi_* X}\, \xi_* Y = \Big(\nabla_X Y + \tfrac{1}{2} R(\xi, \nabla_X \xi) Y + \tfrac{1}{2} R(\xi, \nabla_Y \xi) X\Big)^h + \Big(\nabla_X \nabla_Y \xi \quad \frac{1}{2} R(X,Y)\xi\Big)^v.$$

Therefore, $\xi(M^n)$ is totally geodesic if and only if

$$\big\langle \nabla_X \nabla_Y \xi \quad \frac{1}{2} R(X,Y)\xi, Z\big\rangle + \big\langle \nabla_X Y + \frac{1}{2} R(\xi, \nabla_X \xi) Y + \frac{1}{2} R(\xi, \nabla_Y \xi) X, A^t_\xi Z\big\rangle = 0$$

or equivalently

$$\big\langle \nabla_X \nabla_Y \xi \quad \frac{1}{2} R(X,Y)\xi + A_\xi \;\; \nabla_X Y + \frac{1}{2} R(\xi, \nabla_X \xi) Y + \frac{1}{2} R(\xi, \nabla_Y \xi) X\big), Z \big\rangle = 0.$$

Since $Z \in \xi^\perp$, we can rewrite the letter equation as

$$\nabla_X \nabla_Y \xi \quad \frac{1}{2} R(X,Y)\xi + A_\xi \;\; \nabla_X Y + \frac{1}{2} R(\xi, \nabla_X \xi) Y + \frac{1}{2} R(\xi, \nabla_Y \xi) X\big) = \rho\,\xi,$$

where ρ is some function. Finally, remark that

$$R(X,Y)\xi = \nabla_X \nabla_Y \xi \quad \nabla_Y \nabla_X \xi \quad \nabla_{[X,Y]}\xi$$

and after substitution we get

$$\frac{1}{2} \;\; \nabla_X \nabla_Y \xi + \nabla_Y \nabla_X \xi \quad \nabla_{\nabla_X Y} \xi \quad \nabla_{\nabla_Y X} \xi\big) + \\ \frac{1}{2} A_\xi \;\; R(\xi, \nabla_X \xi) Y + R(\xi, \nabla_Y \xi) X\big) = \rho\,\xi.$$

Taking into account (1), (2) and (3) we can write

$$Hess_\xi(X,Y) + A_\xi Hm_\xi(X,Y) = \rho\,\xi.$$

Multiplying the equation above by ξ, we can find easily $\rho = \;\; \langle A_\xi X, A_\xi Y\rangle$. So, finally

$$Hess_\xi(X,Y) = A_\xi Hm_\xi(X,Y) + \langle A_\xi X, A_\xi Y\rangle\,\xi$$

which completes the proof.

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