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On isometric immersions of 3-dimensional geometries $\widetilde{SL}_2$, Nil, Sol into 4-dimensional space of constant curvature

Abstract(paper to be published in the Ukrain. Mathem. Journal.)

We prove nonexistence of isometric immersions of geometries Nil^3, $\widetilde{SL}_2$ into the 4-dimensional space M_c^4 of constant curature c. The geometry Sol^3 does not admit isometric immersion to M_c^4 for $c \neq -1$ and we find analytic isometric immersion of it into the hyperbolic space $H^4(-1)$.

We study the question of isometric immersion of 3-dimensional geometries of Thurston $\widetilde{SL}_2, Nil, Sol$ [1,p.112] into 4-dimensional space M_c^4 of constant sectional curature c. This problem analiticaly reduces to investigation of solvability the Gauss-Codazzi system in every of three cases, where metric tensor g_{ij} and Riemann tensor are known values and six unknown components of the second fundamental form b_{ij} must satisfy to the system of six Gauss and eight Codazzi equations.

Since this system is overdeterminated, it is natural to expect the negative answer to the question of isometric immersion. Really, it is turned out, that's right for $\widetilde{SL}_2$ and Nil, but in the case of Sol exists the isometric immersion of this space into 4-dimensional hyperbolic space H^4 of constant curvature -1. Nonimmersibility Nil into 4-dimensional euclidean space was proved in [2] by different way. Note, that we calculate components of the Riemann tensor R_{ijkl} and Cristoffel symbols Γ^i_{jk} with the aid of the system Maple.

1. Nonimmersibility of Nil^3 into M_c^4. Threedimensional geometry Nil^3 is the real Lie group with the law of multiplication : $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$, endowed with the left-invariant riemannian metric $ds^2 = dx^2 + dy^2 + (dz - xdy)^2$ [1, p.123].

Theorem 1. *There is no isometric immersion (of class C^3) of any domain of geometry Nil into the space of constant sectional curvature M_c^4.*

Proof. The system of Gauss equations of isometric immersion Nil into M_c^4 is of the form [3, p.182]

$$R_{ijkl} = b_{ik}b_{jl} - b_{il}b_{jk} + c(g_{ik}g_{jl} - g_{il}g_{jk}). \qquad (1)$$

If we substitute here concrete expressions of components of Riemann tensor, calculated with the aid of Maple we get the following six equations

$$
\begin{aligned}
1) R_{1212} &= \frac{x^2-3}{4} = b_{11}b_{22} - b_{12}^2 + c(1+x^2),\\
2) R_{1313} &= \frac{1}{4} = b_{11}b_{33} - b_{13}^2 + c,\\
3) R_{2323} &= \frac{1}{4} = b_{22}b_{33} - b_{23}^2 + c,\\
4) R_{1213} &= -\frac{x}{4} = b_{11}b_{23} - b_{13}b_{12} - cx,\\
5) R_{1223} &= 0 = b_{12}b_{23} - b_{13}b_{22},\\
6) R_{1323} &= 0 = b_{12}b_{33} - b_{13}b_{23}.
\end{aligned} \tag{2}
$$

Multiplying equation 5) on b_{33}, equation 6) on b_{23} and subtracting,

we obtain $b_{13}(b_{22}b_{33} - b_{23}^2) = 0$. Hence, with regard of equation 3), if $c \neq \frac{1}{4}$, we deduce, that $b_{13} = 0$. Similarly, multiplying equation 5) on b_{23}, and equation 6) on b_{22} and subtracting, we get $b_{12} = 0$. Now we can express from equations 1),2),4) b_{22}, b_{33}, b_{23} by means of b_{11} and substitute into 3). Then we obtain $b_{11}^2 = \frac{c}{c-0.25}x^2 - c - 0.75$, $b_{22} = ((0.25-c)x^2 - c - 0.75)b_{11}^{-1}$, $b_{33} = (0.25-c)b_{11}^{-1}$, $b_{23} = -(0.25-c)xb_{11}^{-1}$. Then we use Codazzi equation $b_{12,3} = b_{13,2}$, which is reduced to the form (because nonzero Cristoffel symbols are $\Gamma_{22}^1 = -x$,$\Gamma_{23}^1 = -\Gamma_{13}^2 = \frac{1}{2}$, $\Gamma_{12}^2 = -\Gamma_{13}^3 = \frac{x}{2}$, $\Gamma_{12}^3 = \frac{x^2-1}{2}$), $b_{22} + 2xb_{23} + (x^2-1)b_{33} = 0$. If we substitute in this equation b_{ik}, we get absurd. It remain to study separatly the case $c = \frac{1}{4}$. Then the system (2) takes the form

$$
1) -1 = b_{11}b_{22} - b_{12}^2, 2) 0 = b_{11}b_{33} - b_{13}^2, 3) 0 = b_{22}b_{33} - b_{23}^2, \tag{3}
$$

$$
4) 0 = b_{11}b_{23} - b_{13}b_{12}, 5) 0 = b_{12}b_{23} - b_{13}b_{22}, 6) 0 = b_{12}b_{33} - b_{13}b_{23}.
$$

If we multiply equation 4) on b_{12}, and equation 5) on b_{11} and subtract , then we get $b_{13}(b_{12}^2 - b_{11}b_{22}) = 0$, whence with regard of equation 1), we find, that $b_{13} = 0$. Multiplying equation 4) on b_{22}, equation 5) on b_{12}, and subtracting, we get $b_{23} = 0$, Then it is easy to find, that $b_{33} = 0$. Taking into account the Codazzi equation C1) $b_{32,1} = b_{31,2}$, we obtain, that $b_{22} = -b_{11}$. Consider now the following Codazzi equations

C2) $b_{11,2} = b_{12,1}$, или $\frac{\partial b_{11}}{\partial y} = \frac{\partial b_{12}}{\partial x} + \frac{x}{2}b_{12}$,

C3) $b_{22,1} = b_{21,2}$, или $-\frac{\partial b_{12}}{\partial y} = \frac{\partial b_{11}}{\partial x} + \frac{x}{2}b_{11}$.

We can rewrite C2), C3) in the form

C2) $\frac{\partial}{\partial y}\left(e^{\frac{x^2}{4}}b_{11}\right) = \frac{\partial}{\partial x}\left(e^{\frac{x^2}{4}}b_{12}\right)$, C3) $\frac{\partial}{\partial x}\left(e^{\frac{x^2}{4}}b_{11}\right) = -\frac{\partial}{\partial y}\left(e^{\frac{x^2}{4}}b_{12}\right)$.

Hence, the functions $U = e^{\frac{x^2}{4}}b_{12}$ and $V = e^{\frac{x^2}{4}}b_{11}$ are the real and imaginary parts of some analytic function $U + iV$ of variable $x + iy$. But in this case the Gauss equation 1)$1 = b_{11}^2 + b_{12}^2$ contradicts to the harmonicity of the function $ln(U^2 + V^2)$, q.e.d.

2. Nonimmersibility of $\widetilde{SL}_2$ into M_c^4. Threedimensional geometry $\widetilde{SL}_2$ is the universal covering of the group SL_2 with the left-invariant metric. It is pointed out in [1, p.114] that $\widetilde{SL}_2$, can be presented as fibering of unite vectors UH^2 in the tangent bundle over the hyperbolic plane H^2. Let it be $H^2 : (x, y), y > 0$ with the metric $ds^2_{H^2} = \frac{dx^2+dy^2}{y^2}$. Then UH^2 with coordinates x, y, t is the submanifold in the tangent bundle over H^2 with the metric $ds^2_{UH^2} = ds^2_{H^2} + < Dv, Dv >_{H^2}$, where $v = (y\cos t, y \sin t)$ and $Dv^i = dv^i + \Gamma^i_{jk}v^j dx^k$ $i = 1, 2$ (components of absolut differential of vector v in the metric of H^2). Assuming, that $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}, \Gamma^2_{11} = \frac{1}{y}$, we get

$$ds^2_{UH^2} = \frac{dx^2 + dy^2 + (dx + ydt)^2}{y^2}. \tag{4}$$

Theorem 2. *There is no isometric (class C^3) immersion of any domain of geometry $\widetilde{SL}_2$ into the space of constant curvature M_c^4.*

Proof. Write down the system of Gauss equations of isometric immersion of metric (4) into M_c^4

$$\begin{aligned}
1)R_{1212} &= -\frac{3}{2y^4} = b_{11}b_{22} - b_{12}^2 + \frac{2c}{y^4},\\
2)R_{1313} &= \frac{1}{4y^2} = b_{11}b_{33} - b_{13}^2 + \frac{c}{y^2},\\
3)R_{2323} &= \frac{1}{4y^2} = b_{22}b_{33} - b_{23}^2 + \frac{c}{y^2},\\
4)R_{1223} &= -\frac{1}{4y^3} = b_{12}b_{23} - b_{13}b_{22} - \frac{c}{y^3},\\
5)R_{1332} &= 0 = b_{13}b_{32} - b_{12}b_{33}, 6)R_{2113} = 0 = b_{21}b_{13} - b_{23}b_{11}.
\end{aligned} \tag{5}$$

First consider the case $c \neq \frac{1}{4}$. Acting as in the situation of system (2), we obtain $b_{12} = b_{32} = 0$. Then it is possible to find the rest of unknowns : $b_{11} = \frac{-1.5-2c}{y^4 b_{22}}$, $b_{33} = \frac{0.25-c}{y^2 b_{22}}$, $b_{13} = \frac{0.25-c}{y^3 b_{22}}$, $b_{22} = \frac{\sqrt{-1.75-c}}{y^2}$.

Taking into account the Coddazzi equation $b_{12,3} = b_{13,2}$, (with regard of the following values of Cristoffel symbolsof metric (4) : $\Gamma^1_{12} = -\frac{3}{2y}$, $\Gamma^1_{23} = -\Gamma^2_{13} = -\frac{1}{2}$, $\Gamma^2_{22} = -\frac{1}{y}$, $\Gamma^2_{11} = \frac{2}{y}$, $\Gamma^3_{12} = \frac{1}{y^2}$, $\Gamma^3_{23} = \frac{1}{2y}$), we get absurd identity.

Now consider the case $c = \frac{1}{4}$. The system of Gauss equations (5) takes the form

$$1) b_{11}b_{22} - b_{12}^2 = -\frac{2}{y^4}, 2) b_{11}b_{33} - b_{13}^2 = 0,$$

$$3) b_{22}b_{33} - b_{23}^2 = 0, 4) b_{12}b_{23} - b_{13}b_{22} = 0,$$

$$5) b_{13}b_{32} - b_{12}b_{33} = 0, 6) b_{21}b_{13} - b_{23}b_{11} = 0$$

Multiplying equation 4) on b_{11}, equation 6) on b_{12} and summing, with regard of equation 1), we get $b_{13} = 0$. Multiplying equation 4) on b_{12}, equation 6) on b_{22} and summing, we receive $b_{23} = 0$. Then from equations 2),3),5) и 1) derive that $b_{33} = 0$ and it remains only one Gauss equation $1) b_{11}b_{22} - b_{12}^2 = -\frac{2}{y^4}$ and three unknown functions b_{11}, b_{12}, b_{22} of variables x, y, z, wich must satisfy in addition to six Codazzi equations of the following form:

$C1) b_{12,3} = b_{13,2}$, or $\frac{\partial b_{12}}{\partial z} - \frac{1}{2} b_{22} = 0$,
$C2) b_{21,3} = b_{23,1}$, or $\frac{\partial b_{12}}{\partial z} + \frac{1}{2} b_{11} = 0$,
$C3) b_{11,2} = b_{12,1}$, or $\frac{\partial b_{11}}{\partial y} + \frac{3b_{11}}{2y} = \frac{\partial b_{12}}{\partial x} - \frac{2b_{22}}{y}$,
$C4) b_{11,3} = b_{13,1}$, or $\frac{\partial b_{11}}{\partial z} - \frac{1}{2} b_{21} = 0$,
$C5) b_{22,1} = b_{21,2}$, or $\frac{\partial b_{22}}{\partial x} + \frac{1}{2y} b_{12} = \frac{\partial b_{12}}{\partial y}$,
$C6) b_{22,3} = b_{23,2}$, or $\frac{\partial b_{22}}{\partial z} + \frac{1}{2} b_{21} = 0$.

Equations $C7) b_{33,1} = b_{13,3}$ and $C8) b_{33,2} = b_{32,3}$ are satisfied identically. Comparing equations C1) and C2), we find, that $b_{22} = -b_{11}$ and hence the equations C4) and C6) are equivalent. Then we can rewrite equations C3) and C5) in the form :

$C3) \frac{\partial}{\partial y}\left(\frac{b_{11}}{\sqrt{y}}\right) = \frac{\partial}{\partial x}\left(\frac{b_{12}}{\sqrt{y}}\right)$, $C5) \frac{\partial}{\partial y}\left(\frac{b_{12}}{\sqrt{y}}\right) = -\frac{\partial}{\partial x}\left(\frac{b_{11}}{\sqrt{y}}\right)$,

From here we see that the functions $U = \frac{b_{12}}{\sqrt{y}}$ and $V = \frac{b_{11}}{\sqrt{y}}$ must represent the real and imaginary parts of some analityc function of variable $x + iy$, and also from equation 1) it follows that $U^2 + V^2 = \frac{2}{y^5}$. But then the function $ln(U^2 + V^2)$ is not harmonic, q.e.d.

3. Isometric immersion of Sol^3 into M_c^4. Threedimensional geometry Sol is the Lie group with multiplication law

$$(x, y, z)(x', y', z') = (x + e^{-z}x', y + e^{z}y', z + z'),$$

and the left-invariant metric $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$ [1, p.127].

Theorem 3. *1) There is no isometric (class C^3) immersion of any domain of the geometry Sol^3 into the space of constant curvature M_c^4 if $c \neq -1$. 2) There is isometric analytic immersion of geometry Sol^3 into the hyperbolic space $H^4(-1)$, for example,*

$$\begin{pmatrix} x_0 = \sqrt{2}\cosh\left(z + z_0\right) \\ x_1 = \frac{1}{\sqrt{2}}e^{(z+z_0)}\cos\sqrt{2}x \\ x_2 = \frac{1}{\sqrt{2}}e^{(z+z_0)}\sin\sqrt{2}x \\ x_3 = \frac{1}{\sqrt{2}}e^{(-z-z_0)}\cos\sqrt{2}y \\ x_4 = \frac{1}{\sqrt{2}}e^{(-z-z_0)}\sin\sqrt{2}y \end{pmatrix}, \quad (6)$$

where $H^4(-1) = \left((x_0, x_1, x_2, x_3, x_4)| - x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = -1, x_0 > 0\right)$ is the upper half of hyperboloid of pseudoeuclidean space $R^{4,1}$ with the metric $ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$, $z_0 \in R$ - arbitrary constant.

Proof. The system of Gauss equations of isometric immersion Sol^3 into M_c^4 is of the form

$$1)R_{1212} = 1 = b_{11}b_{22} - b_{12}^2 + c,$$
$$2)R_{1313} = -e^{2z} = b_{11}b_{33} - b_{13}^2 + ce^{2z}, \quad (7)$$
$$3)R_{2323} = -e^{-2z} = b_{22}b_{33} - b_{23}^2 + ce^{-2z}, 4)R_{1223} = 0 = b_{12}b_{23} - b_{13}b_{22},$$
$$5)R_{1332} = 0 = b_{13}b_{32} - b_{12}b_{33}, 6)R_{2113} = 0 = b_{21}b_{13} - b_{23}b_{11}.$$

Acting similarly to solution of system (2), we easily get under conditon $c \neq -1$, that $b_{12} = b_{13} = b_{23} = 0$. Then we find the rest of the coefficients of the second fundamental form: $b_{11} = \sqrt{1-c}e^{2z}$,$b_{22} = \sqrt{1-c}e^{-2z}$, $b_{33} = \frac{|1+c|}{\sqrt{1-c}}$. Taking into account the Codazzi equation $b_{11,3} = b_{13,1}$, it is easy to receive absurd conclusion.Hence, if $c \neq -1$ there is no isometric immersion Sol^3 into M_c^4. In the case $c = -1$ the system (7) takes the form

$$1)b_{11}b_{22} - b_{12}^2 = 2, 2)b_{11}b_{33} - b_{13}^2 = 0,$$

$$3)b_{22}b_{33} - b_{23}^2 = 0, 4)b_{12}b_{23} - b_{13}b_{22} = 0,$$

$$5)b_{13}b_{32} - b_{12}b_{33} = 0, 6)b_{21}b_{13} - b_{23}b_{11} = 0.$$

From equtions 4),6), with regard to equation 1), we obtain,that $b_{13} = b_{23} = 0$. Then from equations 1),2),3),5) we find $b_{33} = 0$. Turning to consideration of the system equtions of Codazzi we find, that $b_{12} = 0$, $b_{11} = c_1e^z$,$b_{22} = \frac{2}{c_1}e^{-z}$, (where c_1 is arbitrary constant), do satisfy to all of them. Then according to famous Bonnet theorem, there is isometric immersion Sol^3 into $H^4(-1)$. Afterwards it is not difficult to find the exact form of immersion (6), cited at the theorem.

[1]. Scott P. The geometries of 3-manifolds. Bulletin of the London Mathem. Soc. .15,1983,n.56, p.401-487 (warning: cited pages correspond to russian translation of this article)

[2]. Rivertz H.J. An obstruction to isometric immersion of the threedimensional Heisenberg group into R^4 // Preprint series 1999, Pure Mathematics, n.22, Matematick institut, Univ. Oslo.

[3]. Eisenhart L.P.Riemannian geometry

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