04. Differential Geometry

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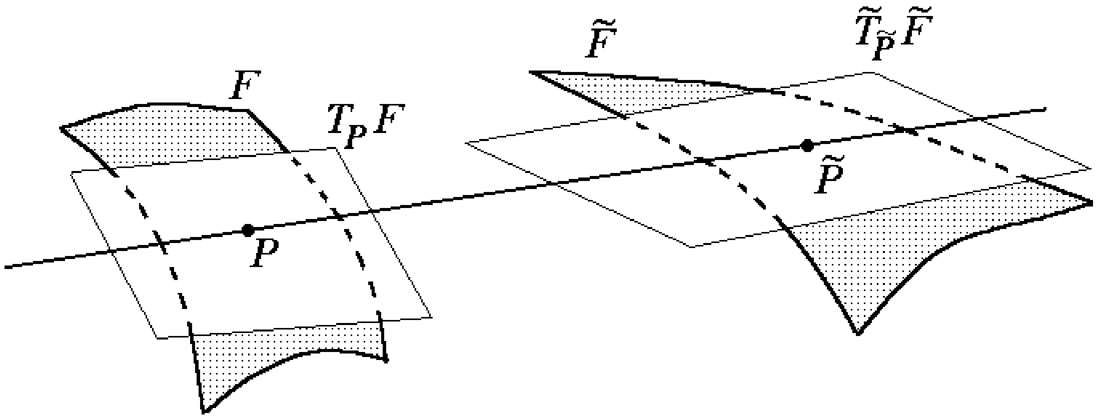
PSEUDO-SPHERICAL CONGRUENCIES IN E^4

Let F, F^* be regular two-dimensional surfaces with point codimension 2 in four-dimensional Euclidean space E^4. *A line congruence* ψ: $F \to F^*$ is a diffeomorphism such that for each point $P \in F$ the straight line joining P and $P^* = \psi(P)$ is a common tangent line for F and F^*. The line congruence ψ is called *pseudo-spherical* if:

1) the distance $|PP^*|$ between P and P^* is equal to l_0, a non-zero constant independent of P.
2) the angle between planes tangent to F and F^* at correspondent points P and P^* is a non-zero constant ω_0 independent of P.

This construction corresponds to the classical definition of pseudo-spherical congruencies for *n*-dimensional submanifolds in *(2n-1)*-dimensional Euclidean space [1].



Generalising classical results by L.Bianchi, G.Darboux and A.V.Backlund, K.Teneblat and C.-L.Terng are demonstrated that *if two n-dimensional submanifolds M, M* in* E^{2n-1} *are relied by a pseudo-spherical congruence, then both M and M* have constant negative sectional curvature*. Moreover, *an arbitrary submanifold* M^n *of constant negative sectional curvature in* E^{2n-1} *admits a large continuous family of different pseudo-spherical congruencies* [1].

Submanifolds of constant negative sectional curvature in Euclidean space are usually referred to as pseudo-spherical. A pseudo-spherical congruence $M \to M^*$ in E^{2n-1}, when viewed as a transformation of n-dimensional submanifolds, is called a *Backlund transformation of pseudo-spherical submanifolds*.

The presented geometric construction was of great importance for the soliton theory. Now it is well-known that pseudo-spherical n-dimensional submanifolds in E^{2n-1} are described by solutions of some special system of non-linear PDEs generalising the classical sine-Gordon equation $z_{xy} = k \sin z$. The geometric Backlund transformation of pseudo-spherical submanifolds may be interpreted as some particular differential relations between corresponding solutions of the mentioned system of non-linear PDEs. These relations posses various interesting properties, whose studying generated important ideas for the theory of solitons. So, actually the pseudo-spherical n-dimensional submanifolds in E^{2n-1} present one of the most used examples of integrable systems [2].

As for n-dimensional pseudo-spherical submanifolds in E^m with $m>2n-1$, an appropriate theory of Backlund transformations is not developed yet. The studying of pseudo-spherical congruencies between two-dimensional surfaces in E^4 was initiated by Yu.Aminov and A.Sym in [3], later some results were obtained by the author: it was demonstrated that the situation here has to be rather different from the classical one.

In order to exclude the particular class of surfaces in three-dimensional affine planes in E^4, which can be treated like to surfaces in E^3, we will consider surfaces in E^4 with two-dimensional first normal space only. Every such surface in E^4 does not belong to any E^3 in E^4, moreover it has a non-degenerate Gauss image.

First of all, if a surface F^2 in E^4 admits a line congruence, then from local point of view there are two possibilities. Either F is a Cartan surface, i.e. it has a unique net of conjugate curves, and the line congruence is determined by straight lines tangent to one family of conjugate curves. Or F is foliated by asymptotic curves and the line congruence is determined by straight lines tangent to these asymptotic curves. So, a generic surface F^2 with two-dimensional first normal space in E^4 admits at most *two* different line congruencies. This conclusion leads to some crucial differences between three- and four-dimensional cases. Note that line congruencies for Cartan surfaces and conjugate nets are widely discussed in [2].

On one hand, some classical statements still valid for surfaces in E^4.

Theorem 1. *Let* ψ*:* $F \to F^*$ *be a pseudo-spherical congruence between two-dimensional surfaces in four-dimensional Euclidean space* E^4*. Then* F *and* F^* *are surfaces of constant negative Gauss curvature* $K=-\sin^2\omega_0 / l_0^2$.

On the other hand, a two-dimensional surface F with constant negative Gauss curvature and with two-dimensional first normal space in E^4 admits *at most two* different pseudo-spherical congruencies, contrary to the classical case. Moreover, there are some reasons to conjecture that a generic two-dimensional surface F with constant negative Gauss curvature in E^4 does not admit pseudo-spherical congruencies. One reason is that if a surface with constant negative Gauss curvature in E^4 admits a pseudo-spherical congruence, then it has to be a Cartan surface. However, it is very possible that in E^4 there exist pseudo-spherical surfaces without conjugate nets, actually it is an open question. As for Cartan surfaces with constant negative Gauss curvature in E^4, every such surface admits at most two different pseudo-spherical congruencies, so an interesting open problem is to find and to describe Cartan surfaces with constant negative Gauss curvature in E^4 which admit either exactly one or exactly two pseudo-spherical congruencies. Some partial answer are discussed below.

Bianchi transformations

Similarly to the classical case, the pseudo-spherical congruencies $F \to F^*$ in E^4 with $\omega_0 = \pi/2$ are referred to as the Bianchi congruencies. The Bianchi congruencies are closely related to the Bianchi transformations of pseudo-spherical surfaces in E^4 discussed by Yu.Aminov and A.Sym [3]. It was demonstrated that a generic surface of constant negative Gauss curvature in E^4 does not admit Bianchi congruencies. Besides the pseudo-spherical surfaces in E^4 that admit Bianchi congruencies form a particular class of surfaces [3].

One way to study the Bianchi congruencies is based on horocyclic coordinates. Namely, let F be a surfaces in E^4 with Gauss curvature $K = -1$ paramete-rised by horocyclic coordinates u, v. The metric form of F^2 is $ds^2 = du^2 + e^{2u} dv^2$, the coordinate lines $v=const$ are geodesics, the coordinate lines $u=const$ are horocircles. Consider a transformation ψ: $F \to F^*$ presented by the following relation:

$$\rho^* = \rho - \partial_u \rho, \tag{BT}$$

where $\rho^*(u,v)$ and $\rho(u,v)$ denote the position-vectors of F and F^* respectively. The mapping ψ is called a Bianchi tranformation [3]. Contrary to the classical case, when a similar mapping transforms a pseudo-spherical surface in E^3 with $K=-1$ into another pseudo-spherical surface in E^3 with $K=-1$, the Gauss curvature of F^* under consideration is not constant. The principal reason is that now the analysed map ψ is not a line congruence. However, if ψ is additionally supposed to be a line congruence then it is a Bianchi congruence ($\omega_0 = \pi/2,\ l_0=1$) and F^* is a pseudo-spherical surface with $K=-1$.

The coefficients of fundamental forms of F^2 with respect to the applied horocyclic coordinates have to satisfy some complicated particular system of non-linear PDEs resulted from the Gauss-Codazzi-Ricci equations for F and from the assumption that ψ is a line congruence. It was demonstrated in [3] that generically this system is not satisfied. Thus the pseudo-spherical surfaces in E^4, whose Bianchi transformation (BT) are Bianchi congruencies, form a particular class of surfaces.

Note that a Bianchi congruence always presents a Bianchi transformation. In fact, if $\psi\colon F \rightarrow F^*$ is supposed to be a Bianchi congruence, when F and F^* are pseudo-spherical by Theorem 1. Next one can prove that there exist a horocyclic coordinate system u, v on F such that the position-vectors of F and F^* are relied by (B), so ψ is a Bianchi tranformation.

Thus it seems more natural to study the Bianchi congruencies rather than the Bianchi tranformations. In our opinion, this way corresponds better to original ideas by L.Bianchi, G.Darboux and A.V.Backlund.

Without loss of generality we may suppose that *F* is a Cartan surface, i.e. *F* contains a unique net of conjugate lines (conjugate with respect to the second fundametal forms of *F*). Let *x, y* stand for the corresponding coordinate system on *F*, i.e. coordinate lines *x=const* and *y=const* form the mentioned conjugate net on *F*. Our main results are two following statement.

Theorem 2. *Let F be a Cartan surface in* E^4. *Let* ψ: $F \to F^*$ *be a Bianchi congruence* ($\omega_0 = \pi/2$, $l_0=1$). *Then F and F* are pseudo-spherical surfaces with Gauss curvature K=-1. Moreover, a unique conjugate net x,y on F is nowhere orthogonal and the coordinate lines y=const of this conjugate net are presented by parallel geodesics with horocyclic orthogonal trajectories. In the general situation F* is also a Cartan surface, the lines x=const and y=const on F* form a conjugate net, the lines x=const are presented by parallel geodesics with horocyclic trajectories. If* $\rho(x,y)$ *is the position-vector of F, when the position-vector* $\rho^*(x,y)$ *of F* has the following form:*

$$\rho^* = \rho - \quad \partial_x\rho, \qquad \text{(BC)}$$

where   stand for the Chrystoffel symbols of F.

Thus, if a pseudo-spherical Cartan surface in E^4 admits a Bianchi congruence then its conjugate net must have particular properties relied to the intrinsic geometry of F. It turns out that this necessary condition is also sufficient.

Theorem 3. *Let F be a Cartan surface in* E^4 *with Gauss curvature K=-1. Suppose that a unique conjugate net x,y on F is nowhere orthogonal and the coordinate lines y=const of this conjugate net are presented by parallel geodesics with horocyclic orthogonal trajectories. Then a map* $\psi: F \rightarrow F^*$ *represented in terms of position-vectors by* (BC) *is a Bianchi congruence.*

Moreover one can demonstrate that a pseudo-spherical surface *F* in E^4 satisfies the conditions of Theorem 3 if and only if the fundamental forms of *F* can be written in terms of some coordinates *x, y* on *F* as follows:

the first fundamental form

$$I = (d\varphi)^2 + e^{2\varphi} (dy)^2$$

the second fundamental forms

$$II^1 = \partial_x\varphi\, e^{-\varphi}(dx)^2 - \partial_x\varphi\, e^{3\varphi}(dy)^2,$$
$$II^2 = Pe^{\varphi}(dy)^2,$$

the normal connection

$$\mu_1=Q,\ \mu_2=0.$$

Here $\varphi(x,y)$, $P(x,y)$, $Q(x,y)$ – some functions which have to satisfy following system of 3 PDEs (the Gauss-Codazzi-Ricci equations) accompanied by regularity conditions:

$$\partial_{xx} e^{2\varphi} + \partial_{yy} e^{-2\varphi} + 2\,(PQ+1) = 0, \quad \text{(B1)}$$
$$\partial_x P - \partial_x\varphi\, e^{2\varphi}\, Q = 0, \quad \text{(B2)}$$
$$\partial_y Q + \partial_y\varphi\, e^{-2\varphi}\, P = 0, \quad \text{(B3)}$$
$$P\neq 0,\ Q\neq 0,\ \partial_x\varphi > 0,\ \partial_y\varphi > 0. \quad \text{(B4)}$$

Thus, a pseudo-spherical Cartan surface F in E^4 which admits a Bianchi congruence $\psi: F \to F^*$ corresponds to a solution $\{\varphi(x,y),\ P(x,y),\ Q(x,y)\}$ of (B1-B4). As for the fundamental forms of F^*, it turns out that they are expressed in terms of $\varphi(x,y),\ P(x,y),\ Q(x,y)$ as follows:

$$I^* = e^{-2\varphi}(dx)^2 + (d\varphi)^2$$
$$II^{*1} = \partial_y\varphi\, e^{-3\varphi}(dy)^2 - \partial_y\varphi\, e^{\varphi}(dy)^2,$$
$$II^{*2} = Qe^{-\varphi}(dx)^2,$$
$$\mu^*_1 = 0,\ \mu^*_2 = P.$$

Therefore, it is easy to see that the Bianchi congruence $\psi: F \to F^*$ corresponds to the following algebraic transformation of solutions of (B1-B4):

$$\{\varphi(x,y),\ P(x,y),\ Q(x,y)\}$$
$$\updownarrow$$
$$\{-\varphi(-y,-x),\ -Q(-y,-x),\ -P(-y,-x)\}$$

Problem. Is (B1-B3) an integrable system?

Example 1. Let us find a travelling wave solution of (B1-B4): $\varphi(x,y)=f(z)$, $P(x,y)=p(z)$, $Q(x,y)=q(z)$, where $z=ax+by$. One can demonstrate that (B1-B3) are satisfied if and only if

$$p = e^{f} (c_2 + c_1 - 2 c_1 f),$$
$$q = e^{-f} (c_2 - c_1 - 2 c_1 f),$$

where c_1 and c_2 are constants, and f(z) solves the following equation:

$$\tfrac{1}{2} (a^2 e^{2f} + b^2 e^{-2f})'' + (c_2 - 2 c_1 f)^2 - (c_1)^2 + 1 = 0.$$

The simplest travelling wave solution is following:

$$\varphi(x,y) = \tfrac{1}{2} \ln (2-(x+y)^2 + (2-(x+y)^2)^2-1),$$

$$P(x,y) = 2-(x+y)^2 + (2-(x+y)^2)^2-1 ,$$

$$Q(x,y) = 2-(x+y)^2 - (2-(x+y)^2)^2-1 .$$

Example 2. Let us find a solution of (B1-B4) in the following form: $\varphi(x,y)$, $P=P(\varphi(x,y))$, $Q=Q(\varphi(x,y))$. One can prove that (B1-B3) are satisfied if and only if

$$P = e^{\varphi} (c_1\varphi + c_2),$$
$$Q = e^{-\varphi} (c_1\varphi + c_1 + c_2),$$

where $\varphi(x,y)$ is a solution of the following equation:

$$\partial_{xx} e^{2\varphi} + \partial_{yy} e^{-2\varphi} + 2((c_1\varphi + c_2)^2 + c_1(c_1\varphi + c_2) + 1) = 0.$$

If $c_1=0$, this equation reads as folows:

$$\partial_{xx} e^{2\varphi} + \partial_{yy} e^{-2\varphi} + C = 0,$$

where $C=2((c_2)^2 + 1)$

References

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