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STABILITY OF ASYNCHRONOUS STATES OF SPIKING NEURONS

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We consider a system of N spiking neurons with random synaptic connections which take values ± 1 with equal probability. The resulting system of equations has a stationary solution equal to 1 for the fraction of neurons having potential x at time t. This solution describes an asynchronous state. We study the stability of such a state in a perturbative way and find a threshold for the parameters of the model such that for values larger than this threshold the stationary asynchronous state is stable otherwise it is unstable. In other terms the stability of the asynchronous state holds only for relatively small random perturbations.

Keywords: spiking neurons; stability; stochastic differential equations.

1. Introduction and Main Results

The synchronized behaviour of systems of neurons is a central issue for the existence of biological organisms. One example is the Parkinson's disease which takes place mostly for the synchronized activity of a particular system of neurons. The stability of synchronous and asynchronous state of activity of neurons is as important as the existence of such states. One way of treating the Parkinson's disease is to send a bipolar electric impulse to the firing neurons which makes the synchronized system unstable [a].

The question of synchronization of activity states of neurons has bean treated by means of models of interacting neurons by various authors (see Refs. 2-13). Some authors considered the influence of excitatory and inhibitory coupling (see Refs. 3, 11, 5,14) but usually it has been considered only one type of interaction: either excitatory or inhibitory coupling. Thus Abbot *et al*[4] have found results on the stability of synchronous and asynchronous state of firing neurons as a function of the sign of the synaptic interaction. In Ref. 3 the authors show, for example, that

[a]Another important example is synchronization of firing characteristics of epileptic activity (see Ref. 1)

if the input to a neurons is inhibitory then the synchronous state is always stable. This unexpected result makes the problem more appealing. In Ref. 4 it is studied the stability of asynchronous state of N interacting IF neurons with excitatory coupling.

An interesting problem is to describe the behavior of the membrane potential in a network of neurons with both excitatory and inhibitory connections possibly randomly distributed. This case is not dealt in the previous literature and we analyze it in the present paper.

We consider the synaptic interaction as a variable $E(t)$ characterizing the time evolution of the current coming from the different neurons and multiply it by a random variable $\tilde{\eta}_i$ which takes values ± 1 with probability one half. This implies that the dynamical equations of the IF system of neurons have a more complicated structure because the right hand side contains a stochastic process generated by the sum of the random contributions of the synaptic inputs.

The state analyzed in this paper is the case of asynchronous firing, i.e. a state with uniform distribution of neural activities. We have to introduce another (stochastic differential) equation for the synaptic interaction with respect to the system treated in Ref. 4.

The stability of the system is derived by studying the dispersion of the stochastic process defining the synaptic interaction. We study the simple case of the evolution equations

$$x_i'(t) = F(x_i) + \xi(t)G(x_i) \tag{1}$$

for $0 < x_i < 1$, $i = 1, \ldots, N$ and $F(x)$ and $G(x)$ being constant. We find that the dispersion of $\xi(t)$ remains always bounded and $\xi(t)$ converges to a stationary stochastic process. On the other hand, there is a threshold for F/G such that above it the dispersion of an arbitrary small perturbation $\tilde{\xi}_1(t)$ remains bounded and below it the dispersion grows up to the infinity. In other words, if the random interaction is small enough with respect to the deterministic part, then the system is stable and the opposite takes place for large random interaction.

The system of equations (1) describes the behavior of a system of N IF neurons . When x_i reaches 1, it emits a spike and resets immediately to 0 in this case $\xi(t)$, the synaptic input to the neurons connected with the neuron reaching the threshold, is incremented as follows:

$$\xi(t) \to \xi(t) + \alpha \frac{\tilde{\eta}_i}{\sqrt{N}} e^{-\alpha(t-t_j)}, \tag{2}$$

where α is a constant characterizing the strength of the synaptic coupling, t_j is the moment of the j-th spike, and $\{\eta_j\}$ are independent random variables for different spikes, assumming values ± 1 with probability $\frac{1}{2}$. Thus the random interaction $\xi(t)$ has the form

$$\xi(t) = \alpha \sum_{t_j \le t} \frac{\tilde{\eta}_j}{\sqrt{N}} e^{-\alpha(t-t_j)}. \tag{3}$$

By a standard way, denoting

$$\begin{aligned}
&y(x) = F_0 \int_0^x \frac{dx'}{F(x')}, \quad F_0^{-1} = \int_0^1 \frac{dx'}{F(x')} \\
&D(y) = F_0 G(x) F^{-1}(x),
\end{aligned}$$

we replace the system (1) by

$$y_i'(t) = F_0 + \xi(t) D(y_i), \tag{4}$$

with the same condition at the point $y = 1$.

Let us define the function

$$\mathcal{N}_N(y,t) = \frac{1}{N} \sum_{i=1}^{N} \theta(y_i(t) - y), \tag{5}$$

where $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. $\mathcal{N}_N(y,t)$ is the ratio of the number of neurons with $y_i \geq y$ and N. Then it is evident that for any smooth function $f(y)$ we have

$$\frac{1}{N} \sum_{i=1}^{N} f(y_i(t)) = \int_0^1 f(y) d\mathcal{N}_N(y,t) \tag{6}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t} \frac{1}{N} \sum_{i=1}^{N} f(y_i(t)) = \frac{1}{N} \sum_{i=1}^{N} f'(y_i(t)) y_i'(t) \\
&= \frac{1}{N} \sum_{i=1}^{N} f'(y_i(t))(F_0 + \xi(t) D(y_i(t)) = \int_0^1 f'(y)(F_0 + \xi(t) D(y)) d\mathcal{N}_N(y,t).
\end{aligned} \tag{7}$$

Let us assume that for any y, t there exists a limit

$$\mathcal{N}(y,t) = \lim_{N \to \infty} \mathcal{N}_N(y,t),$$

this measure is absolutely continuous with respect to the Lebesgue measure and the respective density $\rho(y,t)$ is a smooth function in y, t. Then (6) and (7) give us in the limit $N \to \infty$:

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_0^1 f(y) \rho(y,t) dy = \int_0^1 f'(y)(F_0 + \xi(t) D(y)) \rho(y,t) dy \\
&= -\int_0^1 f(y) \frac{\partial}{\partial y} \left((F_0 + \xi(t) D(y)) \rho(y,t) \right) dy
\end{aligned}$$

and since $f(y)$ is an arbitrary function we obtain the equation

$$J(y,t) = \rho(y,t)(F_0 + \xi(t) D(y)). \tag{8}$$

From (4),(5), we get

$$\frac{\partial}{\partial t} \rho(y,t) = -\frac{\partial}{\partial y} J(y,t), \tag{9}$$

with the periodic boundary conditions for $J(y,t)$: $J(0,t) = J(1,t)$.

With above definitions representation (3) in the limit $N \to \infty$ takes the form

$$\begin{aligned}\xi(t) &= \alpha \int_0^t e^{-\alpha(t-t')} \sqrt{J(1,t)} d\eta(t') \\ &= \alpha \int_0^t e^{-\alpha(t-t')} \sqrt{\rho(1,t')(F_0 + \xi(t')D(1))} d\eta(t'),\end{aligned} \tag{10}$$

where $\eta(t)$ is a standard Gaussian white noise.

We study the simplest case when $D(y)$ does not depend on y. Hence, in view of the above discussion, we have to study the system of equations:

$$\begin{aligned}&\frac{\partial \rho}{\partial t} = -(F_0 + \xi(t)D) \frac{\partial}{\partial y} \rho(y,t) \\ &d\xi(t) = -\alpha \xi(t) dt + \alpha \sqrt{\rho(1,t)(F_0 + \xi(t)D)} d\eta(t) \\ &\rho(0,t) = \rho(1,t).\end{aligned} \tag{11}$$

It is easy to see that this system has the solution

$$\begin{aligned}&\rho(y,t) = 1 \\ &J(y,t) = (F_0 + \xi_0(t)D),\end{aligned} \tag{12}$$

where the random process $\xi_0(t)$ is a solution of the stochastic differential equation

$$d\xi_0(t) = -\alpha \xi_0(t) dt + \alpha \sqrt{F_0 + \xi_0(t)D} d\eta(t). \tag{13}$$

We remark here that it is proven in Lemma 1 that the argument of the square root is always positive.

The main goal of this paper is to study the solution (12)-(13) and its stability with respect to small perturbations.

Theorem 1: The process $\xi_0(t)$ converge in probability, as $t \to \infty$, to the stationary diffusion process with the correlation function

$$R(t-s) = F_0 (2\alpha)^{-1} e^{-\alpha(t-s)}. \tag{14}$$

To study the stability of solution (12) we take ε small and consider

$$\rho(y,t) = 1 + \varepsilon \rho_1(y,t), \quad \xi(t) = \xi_0(t) + \varepsilon \tilde{\xi_1}(t). \tag{15}$$

Then in the first order with respect to ε we obtain the system:

$$\begin{aligned}&\frac{\partial}{\partial t} \rho_1(y,t) = -(F_0 + \xi_0(t)D) \frac{\partial}{\partial y} \rho_1(y,t) \\ &d\xi_1(t) = -\alpha \xi_1(t) dt + \frac{\alpha \xi_1(t)}{2\sqrt{F_0 + \xi_0(t)D}} d\eta(t) + \frac{\alpha}{2} \rho_1(1,t) \sqrt{F_0 + \xi_0(t)D} d\eta(t) \\ &\rho_1(0,t) = \rho_1(1,t).\end{aligned} \tag{16}$$

Definition 2: We say that some solution (ρ, J, ξ) of the system of equations (8)-(10) is stable, if for any perturbation $\rho_1(y,0)$ of the initial conditions, the perturbation terms $\rho_1(y,t)$, $\xi_1(t)$ satisfy the inequalities:

$$\begin{aligned}E\Big\{ \int_0^1 |\rho_1(y,t)|^2 dy \Big\} &\le \; const\, E\Big\{ \max_y |\rho_1(y,0)|^2 \Big\} \\ E\Big\{ \xi_1^2(t) \Big\} &\le \; const\, E\Big\{ \max\nolimits_y |\rho_1(y,0)|^2 \Big\}.\end{aligned} \tag{17}$$

Here and below we denote by the symbol $E\{\dots\}$ the averaging with respect to all random parameters of the problem.

Theorem 3: Solution (12) is stable if $\frac{F_0}{\alpha D^2} > \frac{1}{4}$ and unstable if $\frac{F_0}{\alpha D^2} < \frac{1}{4}$.

2. Proofs

Proof of Theorem 1: By using the representation

$$\xi(t) = e^{-\alpha t}\xi_0(0) + \alpha \int_0^t e^{-\alpha(t-t')}\sqrt{(F_0 + \xi_0(t')D)}d\eta(t'), \tag{18}$$

one can get easily that

$$E\{\xi_0(t)\} = e^{-\alpha t}E\{\xi_0(0)\},$$

so we get that the mean value of $\xi_0(t)$ tends to zero exponentially. Now for $t > s$ let us write

$$\xi_0(t) = \xi_0(s)e^{-\alpha(t-s)} + \alpha \int_s^t e^{-\alpha(t-t')}\sqrt{F_0 + \xi_0(t')D}d\eta(t').$$

Multiplying this system by $\xi_0(s)$ and taking the expectation, we get

$$E\{\xi_0(t)\xi_0(s)\} = E\{\xi_0(s)\xi_0(s)\}e^{-\alpha(t-s)}, \tag{19}$$

because, as usually in the theory of stochastic integrals,

$$E\left\{\xi_0(s)\int_s^t e^{-\alpha(t-t')}\sqrt{F_0 + \xi_0(t')D}d\eta(t')\right\}$$
$$= E\left\{\int_s^t e^{-\alpha(t-t')}\sqrt{F_0 + \xi_0(t')D}d\eta(t')\,\Big|\,\xi_0(s)\right\}E\left\{\xi_0(s)\right\} = 0.$$

Thus we are left to study $E\{\xi_0^2(s)\}$. Using again formula (18), by the standard way we get

$$\begin{aligned}
E\{\xi_0(s)\xi_0(s)\} &= E\{\xi_0(0)\xi_0(0)\}e^{-2\alpha s} + E\left\{\int_0^s e^{-2\alpha(s-t')}(F_0 + D\xi_0(t'))dt'\right\}\\
&= E\{\xi_0(0)\xi_0(0)\}e^{-2\alpha s} + \int_0^s e^{-2\alpha(s-t')}(F_0 + DE\{\xi_0(t')\})dt'\\
&= E\{\xi_0(0)\xi_0(0)\}e^{-2\alpha s} + \int_0^s e^{-2\alpha(s-t')}(F_0 + De^{-\alpha t'}E\{\xi_0(0)\})dt'\\
&= E\{\xi_0(0)\xi_0(0)\}e^{-2\alpha s} + \frac{F_0}{2\alpha}(1 - e^{-2\alpha s}) + \frac{D}{\alpha}E\{\xi_0(0)\}(e^{-\alpha s} - e^{-2\alpha s}).
\end{aligned}$$

□

To prove Theorem 3 we need some additional information about $\xi_0(t)$.

Lemma 4: *Denote* $\gamma = 2(\alpha D^2)^{-1}F_0$. *Then*

$$\mathcal{E}(t) = E\left\{(F_0 + D\xi_0(t))^{-1}\right\} \le C < \infty, \quad (if\ \gamma - 1 > 0)$$

and $\mathcal{E}(t) = \infty$, *if* $\gamma - 1 < 0$.

Proof: To simplify formulas below we make the change of variables

$$\begin{array}{l}\beta=(\alpha D^2)^{-1},\quad \tilde t=(\alpha D)^2 t,\\ \tilde\xi(\tilde t)=F_0+D\xi(\tilde t/(\alpha D)^2),\quad \tilde\rho(\tilde t)=\rho(\tilde t/(\alpha D)^2),\quad \tilde\eta(\tilde t)=\alpha D\cdot\eta(\tilde t/(\alpha D)^2)\end{array}\tag{20}$$

Then $\tilde\eta(\tilde t)$ is again a standard white noise with respect to $\tilde t$. But to simplify the notations below we write t instead of $\tilde t$ and η instead of $\tilde\eta$. Then (11) takes the form

$$\begin{array}{l}\frac{\partial\tilde\rho}{\partial t}=-(\alpha D)^{-2}\tilde\xi\frac{\partial}{\partial y}\tilde\rho(y,t)\\ d\tilde\xi(t)=-\beta(\tilde\xi(t)-F_0)dt+\sqrt{\tilde\rho(1,t)\tilde\xi(t)}d\eta(t)\\ \tilde\rho(0,t)=\tilde\rho(1,t)\end{array}\tag{21}$$

and (13) takes the form

$$d\tilde\xi_0(t)=-\beta(\tilde\xi_0(t)-F_0)dt+\sqrt{\tilde\xi_0(t)}d\eta(t),\tag{22}$$

where $\tilde\xi_0(t)=F_0+D\xi_0(t/(\alpha D)^2)$.

Let $p(x,t\,|\,y,s)$ be the probability density of transition from y at time s to x at time t of the diffusion process generated by the solutions of the equation (22):

$$\mathrm{Prob}\Big\{\tilde\xi_0(t)\in\Delta\subset\mathbf{R},\,|\,\tilde\xi_0(s)=y\Big\}=\int_\Delta p(x,t\,\Big|\,y,s)dx.\tag{23}$$

Then one can write the direct Kolmogorov equation or the Fokker-Planck equation (see, e.g. Ref. 16) for the function $p(x,t|y,s)$:

$$\frac{\partial}{\partial t}p=\beta\frac{\partial}{\partial x}\Big((x-F_0)p\Big)+\frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(xp\Big).\tag{24}$$

with the initial condition

$$p(x,s\,|\,y,s)=\delta(x-y).\tag{25}$$

Taking the Fourier transform

$$\hat p(k,t,y,s)=\int e^{ikx}p(x,s\,|\,y,s)dx,$$

we obtain from (24) the first order differential equation for $\hat p(k,t)$

$$\frac{\partial}{\partial t}\hat p(k,t,y,s)=(\frac{ik^2}{2}-\beta k)\frac{\partial}{\partial k}\hat p(k,t,y,s)+i\beta F_0k\hat p(k,t,y,s)\tag{26}$$

with the initial and the boundary conditions

$$\hat p(k,s,y,s)=e^{iky},\quad \hat p(k,t,y,s)\to 0,\ k\to\pm\infty.\tag{27}$$

Let us make the change of variables

$$\begin{array}{l}\tilde k=\beta^{-1}\log\frac{k}{k+2i\beta}\Leftrightarrow k=-\frac{2\beta ie^{\beta\tilde k}}{e^{\beta\tilde k}-1},\\ \hat p(k,t)=e^{c(\tilde k,t)}.\end{array}$$

Then we obtain the standard linear equation of the first order with respect to the function $c(\tilde{k}, t)$:

$$\frac{\partial c}{\partial t} + \frac{\partial c}{\partial \tilde{k}} = 2\beta^2 F_0 \frac{e^{\beta\tilde{k}}}{e^{\beta\tilde{k}} - 1}.$$

Therefore, by a standard method we get:

$$\begin{aligned} c(\tilde{k}, t) &= c_0(\tilde{k} - t + s) + \int_s^t 2\beta^2 F_0 \frac{e^{\beta(\tilde{k}-t+t')}}{e^{\beta(\tilde{k}-t+t')} - 1} dt' \\ &= c_0(\tilde{k} - t + s) + 2\beta F_0 \Big(\log(e^{\beta\tilde{k}} - 1) - \log(e^{\beta(\tilde{k}-t+s)} - 1) \Big), \end{aligned} \tag{28}$$

where the function c_0 should be found from the initial conditions:

$$c_0(\tilde{k}) = iyk = y \cdot \frac{2\beta e^{\beta\tilde{k}}}{e^{\beta\tilde{k}} - 1}.$$

So, we have from (28)

$$c(\tilde{k}, t) = y \cdot \frac{2\beta e^{\beta(\tilde{k}-t+s)}}{e^{\beta(\tilde{k}-t+s)} - 1} + 2\beta F_0 \Big(\log(e^{\beta\tilde{k}} - 1) - \log(e^{\beta(\tilde{k}-t+s)} - 1) \Big).$$

Now, replacing $e^{\beta\tilde{k}}$ by $\dfrac{k}{k + 2i\beta}$ and putting $\hat{p}(k, t) = e^{c(k,t)}$, we obtain :

$$\hat{p}(k, t, y, s) = (2i\beta)^{-2\beta F_0} \frac{\lambda_1^{2\beta F_0}(\tau)}{(k + i\lambda_1(\tau))^{2\beta F_0}} \exp\left\{ - \frac{k\lambda_2(\tau)y}{k + i\lambda_1(\tau)} \right\}, \tag{29}$$

with

$$\lambda_1(\tau) = \frac{2\beta}{1 - e^{-\beta\tau}}, \quad \lambda_2(\tau) = \frac{2\beta e^{-\beta\tau}}{1 - e^{-\beta\tau}}.$$

where $\tau = t - s$.

We remark that one can get also another solution by separating the variables of (26) which satisfies the initial condition in t but doesn't go to 0 for $k \to \infty$.

To find $p(x, t \,|\, y, s)$ we have to take the inverse Fourier transform of (29):

$$\begin{aligned} p(x, t \,|\, y, s) = &\frac{\lambda_1^{2\beta F_0}(\tau) e^{-\lambda_1(\tau)x - \lambda_2(\tau)y}}{2\pi(2i\beta)^{2\beta F_0}} \\ &\times \int dk (k + i\lambda_1(\tau))^{-2\beta F_0} \exp\left\{ \frac{i\lambda_1(\tau)\lambda_2(\tau)y}{k + i\lambda_1(\tau)} - ix(k + i\lambda_1(\tau)) \right\}, \end{aligned} \tag{30}$$

We remark first that, by the standard method, i.e., taking the integral over the contour $|k| = R$, $\Im k \geq 0$ with $R \to \infty$ instead of the real line, we get that for $x < 0$ $p(x, t \,|\, y, s) = 0$.

To take the integral in (30) for $x > 0$ we observe that, if we define the function

$$w(z) = z^{-\nu} \int dk (k + i\lambda)^{-\nu-1} \exp\{ia(k + i\lambda)^{-1} - iz^2(k + i\lambda)\}, \tag{31}$$

then

$$w'' = (\frac{\nu^2}{z^2} + 4a)w - \frac{1}{z}w' \Rightarrow z^2w'' + zw' - (\nu^2 + 4az^2)w = 0. \tag{32}$$

Indeed, denoting for simplicity

$$\mathcal{I}_\nu = \int dk(k+i\lambda)^{-\nu}\exp\{ia(k+i\lambda)^{-1} - iz^2(k+i\lambda)\}, \quad (\nu = \nu-1, \nu, \nu+1),$$

we have

$$\begin{aligned} &w = z^{-\nu}\mathcal{I}_{\nu+1}, \\ &w' = -\nu z^{-\nu-1}\mathcal{I}_{\nu+1} - 2iz^{-\nu+1}\mathcal{I}_\nu, \\ &w'' = \nu(\nu+1)z^{-\nu-2}\mathcal{I}_{\nu+1} + 2i(2\nu-1)z^{-\nu}\mathcal{I}_\nu - 4z^{-\nu+2}\mathcal{I}_{\nu-1}. \end{aligned} \tag{33}$$

From the second line we get

$$-2iz^{-\nu+1}\mathcal{I}_\nu = w' + \nu z^{-1}w. \tag{34}$$

Besides, integrating by parts, we obtain

$$\begin{aligned} &\int dk(-\frac{ia}{(k+i\lambda)^2} - iz^2)\frac{\exp\{ia(k+i\lambda)^{-1} - iz^2(k+i\lambda)\}}{(k+i\lambda)^{\nu-1}} \\ &= (\nu-1)\int dk\frac{\exp\{ia(k+i\lambda)^{-1} - iz^2(k+i\lambda)\}}{(k+i\lambda)^\nu} \\ &\Rightarrow z^2\mathcal{I}_{\nu-1} = -a\mathcal{I}_{\nu+1} + i(\nu-1)\mathcal{I}_\nu \\ &\Rightarrow z^{-\nu+2}\mathcal{I}_{\nu-1} = -aw + i(\nu-1)z^{-\nu}\mathcal{I}_\nu \\ &= -aw - \frac{\nu-1}{2z}(w' + \frac{\nu}{z}w). \end{aligned} \tag{35}$$

Substituting (35) and (34) in (33) we obtain (32). Thus (see Ref. 15), $w(z) = I_\nu(2a^{1/2}z)$ (up to the multiplicative constant), where

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{z^{2m}}{2^{2m}m!\Gamma(m+\nu+1)}$$

is the modified Bessel function.

Now, using representation (31) with $a = \lambda_1(\tau)\lambda_2(\tau)y$, $z^2 = x$, $\nu = 2\beta F_0 - 1 = \gamma - 1$, one can obtain from (30) that for $x > 0$

$$p(x,t\,|\,y,s) = C\lambda_1(t)e^{-\lambda_1(t)x-\lambda_2(t)y}\left(\frac{\lambda_1(t)x}{\lambda_2(t)y}\right)^{(\gamma-1)/2} I_{\gamma-1}\left(2\sqrt{\lambda_1(t)\lambda_2(t)xy}\right), \tag{36}$$

where C is some constant. One can see that this solution behaves like $Cx^{\gamma-1}$, as $x \sim 0$ and that $\tilde{\xi}_0(t)$ takes positive values. □

Proof of Theorem 3: Now we seek the solution of (21) in the form $\tilde{\xi}(t) = \tilde{\xi}_0(t) + \varepsilon\tilde{\xi}_1(t)$ and $\rho_1(y,t) = 1 + \varepsilon\tilde{\rho}(t)$. Let us substitute these expressions in (21) and take the first order terms with respect to ε. Then, since the first equation of (21) has the solution

$$\tilde{\rho}_(y,t) = \rho(y - (\alpha D)^{-2}\sigma, 0) = 1 + \epsilon\rho_1(y - (\alpha D)^{-2}\sigma, 0) + o(\varepsilon), \tag{37}$$

with $(\sigma(t) = \int_0^t \tilde{\xi}_0(t')dt'$, we get the system of three stochastic differential equations:

$$\begin{aligned}
&d\tilde{\xi}_0(t) = -\beta(\tilde{\xi}_0(t) - F_0)dt + \sqrt{\tilde{\xi}_0(t)}d\eta(t)\\
&d\sigma(t) = \tilde{\xi}_0(t)dt\\
&d\tilde{\xi}_1(t) = -\beta\tilde{\xi}_1(t)dt + \frac{\tilde{\xi}_1(t)}{2\sqrt{\tilde{\xi}_0(t)}}d\eta(t) + \frac{1}{2}\tilde{\rho}_1(\sigma(t))\sqrt{\tilde{\xi}_0(t)}d\eta(t),
\end{aligned} \tag{38}$$

where

$$\tilde{\rho}_1(\sigma) = \rho_1(1 - (\alpha D)^{-2}\sigma, 0) \tag{39}$$

The Kolmogorov equation in this case is too complicated, but using the fact, that the third equation is linear with respect to $\tilde{\xi}_1(t)$, we can solve this equation directly. First we solve the homogeneous version of the equation.

$$d\zeta_0(t) = -\beta\zeta_0(t)dt + \frac{\zeta_0(t)d\eta(t)}{2\sqrt{\tilde{\xi}_0(t)}}. \tag{40}$$

We seek $\zeta_0(t)$ in the form

$$\zeta_0(t) = \tilde{\xi}_0^{1/2}(t)e^{f(t)}, \tag{41}$$

where $E\{|f'(t)|^\nu\} < \infty$ for some small enough ν. Then, by using the Ito formula (see Ref. 16), one can write

$$\begin{aligned}
d\zeta_0(t) &= \frac{bf d\tilde{\xi}_0(t)}{2\tilde{\xi}_0^{1/2}(t)}e^{f(t)} - \frac{\tilde{\xi}_0(t)}{8\tilde{\xi}_0^{3/2}(t)}e^{f(t)}dt + \tilde{\xi}_0^{1/2}(t)f'(t)e^{f(t)}dt\\
&= \frac{1}{2}e^{f(t)}d\eta(t) + \tilde{\xi}_0^{1/2}(t)e^f\Big(- \frac{\beta(\tilde{\xi}_0(t) - F_0)}{2\tilde{\xi}_0(t)} - \frac{1}{8\tilde{\xi}_0} + f'(t)\Big)dt.
\end{aligned}$$

Here we have used the first equation of (38) for $d\xi_0(t)$. Now we compare the coefficients of $d\eta(t)$ and dt in the above expression for $d\zeta_0(t)$ with those in (40). It is evident that the coefficients of $d\eta(t)$ coincide, because in view of (41) $\frac{1}{2}e^f = \frac{1}{2}\zeta_0\xi_0^{-1/2}$. Furthermore, comparing the coefficients of dt, we get

$$\begin{aligned}
&\tilde{\xi}_0^{1/2}(t)e^f\Big(- \frac{\beta(\tilde{\xi}_0(t) - F_0)}{2\tilde{\xi}_0(t)} - \frac{1}{8\tilde{\xi}_0} + f'(t)\Big) = -\beta\tilde{\xi}_0^{1/2}(t)e^f \Rightarrow\\
&f'(t) = -\frac{\beta}{2} - \frac{2\gamma - 1}{8\tilde{\xi}_0(t)} \quad \Rightarrow f(t) = -\frac{\beta t}{2} - \frac{2\gamma - 1}{8}\int_0^t \frac{dt'}{\tilde{\xi}_0(t')}\\
&\Rightarrow \zeta_0(t) = \tilde{\xi}_0^{1/2}(t)\exp\Big\{ - \frac{\beta t}{2} - \frac{2\gamma - 1}{8}\int_0^t \frac{dt'}{\tilde{\xi}_0(t')}\Big\}
\end{aligned}$$

with γ defined in Lemma 4. Now we seek $\tilde{\xi}_1(t)$ - the solution of (38) in the form:

$$\tilde{\xi}_1(t) = \zeta_0(t)\zeta_1(t),$$

where $\zeta_1(t)$ is an unknown Markov process with the stochastic differential

$$d\zeta_1(t) = u(t)dt + v(t)d\eta(t).$$

Here $u(t)$ and $v(t)$ are unknown coefficients, which one can find, using the Ito formula. Indeed, by this formula

$$\begin{aligned}
d\tilde{\xi}_1(t) &= \zeta_1(t)d\zeta_0(t) + \zeta_0(t)d\zeta_1(t) + v(t)\frac{\zeta_0(t)}{2\tilde{\xi}_0^{1/2}(t)}dt \\
&= \Big(-\beta\tilde{\xi}_1(t)dt + \frac{\tilde{\xi}_1(t)}{2\tilde{\xi}_0^{1/2}(t)}d\eta(t)\Big) \\
&+\Big(\zeta_0(t)u(t) + \frac{v(t)\zeta_0(t)}{2\tilde{\xi}_0^{1/2}(t)}\Big)dt + v(t)\zeta_0(t)d\eta(t).
\end{aligned}$$

Now by using the third equation in (38) we find

$$\zeta_0(t)u(t) + \frac{v(t)\zeta_0(t)}{2\tilde{\xi}_0^{1/2}(t)} = 0, \quad v(t)\zeta_0(t) = \frac{\alpha}{2}\tilde{\rho}(\sigma(t))\tilde{\xi}_0^{1/2}(t),$$

and so

$$v(t) = \frac{1}{2}\zeta_0^{-1}(t)\tilde{\rho}(\sigma(t))\tilde{\xi}_0^{1/2}(t), \quad u(t) = -\frac{1}{4}\zeta_0^{-1}\tilde{\rho}(\sigma(t)),$$

which gives us

$$\tilde{\xi}_1(t) = -\frac{1}{4}\zeta_0(t)\int_0^t \zeta_0^{-1}(s)\tilde{\rho}(\sigma(s))ds + \frac{1}{2}\zeta_0(t)\int_0^t \zeta_0^{-1}(s)\tilde{\rho}(\sigma(s))\tilde{\xi}_0^{1/2}(s)d\eta(s). \quad (42)$$

Now, one can find easily, that if $2\gamma - 1 > 0$, then we can bound the variance of $\tilde{\xi}_1(t)$ as follows:

$$\begin{aligned}
E\{\tilde{\xi}_1^2(t)\} &= \frac{1}{16}E\Big\{\zeta_0^2(t)\Big[\int_0^t \zeta_0^{-1}(s)\tilde{\rho}(\sigma(s))ds\Big]^2\Big\} \\
&+\frac{1}{4}E\Big\{\zeta_0^2(t)\int_0^t \zeta_0^{-2}(s)|\tilde{\rho}^2(\sigma(s))|\tilde{\xi}_0(s)ds\Big\}.
\end{aligned} \quad (43)$$

Here the second integral satisfies the bound:

$$\begin{aligned}
I_1 &= \zeta_0^2(t)\int_0^t \zeta_0^{-2}(s)|\tilde{\rho}^2(\sigma(s))|\tilde{\xi}_0(s)ds \\
&= \tilde{\xi}_0(t)\int_0^t \exp\Big\{ -\beta(t-s) - \frac{2\gamma-1}{4}\int_s^t \frac{dt'}{\tilde{\xi}_0(t')}\Big\}|\tilde{\rho}^2(\sigma(s))|ds \\
&\le |\max_t \tilde{\rho}|^2\tilde{\xi}_0(t)\int_0^t dse^{-\beta(t-s)} \le |\max_t \tilde{\rho}|^2\tilde{\xi}_0(t)\beta^{-1}.
\end{aligned}$$

And the first integral, by using the Schwartz inequality, can be estimated as

$$
\begin{aligned}
&I_2 = \left|\zeta_0(t)\int_0^t \zeta_0^{-1}(s)\tilde\rho(\sigma(s))ds\right| \\
&\le |\max_t \tilde\rho|\,\tilde\xi_0^{1/2}(t)\int_0^t \exp\Big\{-\frac{\beta}{2}(t-s)-\frac{2\gamma-1}{8}\int_s^t\frac{dt'}{\tilde\xi_0(t')}\Big\}\tilde\xi_0^{-1/2}(s)ds \\
&\le |\max_t \tilde\rho|\,\tilde\xi_0^{1/2}(t)\Big[\int_0^t e^{-\beta(t-s)/2}ds\Big]^{1/2} \\
&\Big[\int_0^t \tilde\xi_0^{-1}(s)\exp\Big\{-\frac{\beta}{2}(t-s)-\frac{2\gamma-1}{4}\gamma\int_s^t\frac{dt'}{\tilde\xi_0(t')}\Big\}ds\Big]^{1/2} \\
&= 4\beta\tilde\xi_0^{1/2}(t)\,|\max_t\tilde\rho|\,(1-e^{-\beta t/2})\cdot I_3^{1/2}.
\end{aligned}
$$

Using the fact that

$$\tilde\xi_0^{-1}(s) = -\frac{d}{ds}\int_s^t\frac{dt'}{\tilde\xi_0(t')},$$

one can integrate by parts in I_3 and obtain

$$
\begin{aligned}
I_3 = &\frac{4}{2\gamma-1}\exp\Big\{-\frac{\beta}{2}(t-s)-\frac{2\gamma-1}{4}\int_s^t\frac{dt'}{\tilde\xi_0(t')}\Big\}\Big|_0^t \\
&-\frac{2\beta}{2\gamma-1}\int_0^t\exp\Big\{-\frac{\beta}{2}(t-s)-\frac{2\gamma-1}{4}\int_s^t\frac{dt'}{\tilde\xi_0(t')}\Big\}ds \\
&\le\frac{4}{2\gamma-1}.
\end{aligned}
$$

Now let us recall that by definition $\tilde\rho(\sigma(t)) = \rho_1(1-(\alpha D)^{-2}\sigma(t),0)$. Therefore

$$|\max_t\tilde\rho| = |\max_y\rho_1(y,0)|.$$

Thus, we obtain from the above estimates that

$$
\begin{aligned}
E\{\tilde\xi_1^2\} &\le \text{ const}\,|\max_y\rho_1(y,0)|^2E\{\tilde\xi_0(t)\} \\
&= \text{ const}\,|\max_y\rho_1(y,0)|^2E\{F_0+D\xi_0(\alpha Dt)\},
\end{aligned}
$$

were we have come back to our initial notations. Now Theorem 1 gives us the statement of Theorem 3 for $2\gamma-1>0$.

Now, if $2\gamma-1<0$, taking $\rho_1(y,0)=e^{ik\pi}$, we get from (43) that

$$E\{\tilde\xi_1^2(t)\} \ge \frac{1}{4}E\Big\{\tilde\xi_0(t)\int_0^t\exp\Big\{-\frac{1}{2}\beta(t-s)+\frac{1-2\gamma}{8}\int_s^t\frac{dt'}{\tilde\xi_0(t')}\Big\}ds\Big\}. \tag{44}$$

Let us choose $\frac{\varepsilon}{1-\varepsilon}<\beta F_0$. Then, using Lemma 4 one can conclude that

$$E\{\tilde\xi_0^{-\varepsilon/(1-\varepsilon)}(t)\}\le C<\infty,\quad E\{\tilde\xi_0^{-1}\}=\infty.$$

Thus, denoting by I_0 the integral in (44) and using the Holder inequality with $p=\varepsilon^{-1}$, $q=(1-\varepsilon)^{-1}$, we obtain

$$
\begin{aligned}
&E\{(\tilde\xi_0^\varepsilon(t)I_0^\varepsilon)\tilde\xi_0^{-\varepsilon}\}\le E^{1/p}\{\tilde\xi_0I_0\}E^{1/q}\{\tilde\xi_0^{-\varepsilon q}\} \\
&\Rightarrow E\{\tilde\xi_0I_0\}\ge\Big(E\{I_0^\varepsilon\}E^{(\varepsilon-1)}\{\tilde\xi_0^{-\varepsilon/(1-\varepsilon)}\}\Big)^{1/\varepsilon}.
\end{aligned}
\tag{45}
$$

But

$$\begin{aligned} I_0 &= \int_0^t ds\, \exp\Big\{ -\frac{1}{2}\beta(t-s) + \frac{1-2\gamma}{8}\int_s^t \frac{dt'}{\tilde{\xi}_0(t')}\Big\} \\ &\geq \int_0^{t/2} ds\, \exp\Big\{ -\frac{1}{2}\beta(t-s) + \frac{1-2\gamma}{8}\int_{t/2}^t \frac{dt'}{\tilde{\xi}_0(t')}\Big\} \\ &\geq \tfrac{1}{2}\beta^{-1}e^{-\beta t/2}\exp\Big\{\frac{1-2\gamma}{8}\int_{t/2}^t \frac{dt'}{\tilde{\xi}_0(t')}\Big\}. \end{aligned}$$

Now we get from (44) and (45) and the Jensen inequality

$$E\{\tilde{\xi}_1^2(t)\} \geq Ce^{-\beta t/2}\exp\Big\{\frac{1-2\gamma}{8}\int_{t/2}^t E\{\tilde{\xi}_0^{-1}(t')\}dt'\Big\} = \infty.$$

Here we have used Lemma 4, according to which $E\{\tilde{\xi}_0^{-1}(t')\} = \infty$ for $\gamma < 1$. □

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