CONVEXITY AND INTERSECTION OF RANDOM SPACES

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Abstract The problem of finding the volume of the intersection of the N dimensional sphere with $p = \alpha N$ random half spaces when α is less than a critical value α_c and when $N, p \to \infty$ is solved rigorously. The asymptotic expression coincides with the one found by E. Gardner ([4]), using non rigorous replica calculations in neural network theory. When α is larger than α_c the volume of the intersection goes to 0 more rapidly than $\exp(-N \,\text{const}\,)$. We use the cavity method. The convexity of the volume and the Brunn Minkowski theorem ([3]) have a central role in the proof.

Keywords: Random systems, Convexity,

1. Introduction

For very large integer N consider the N-dimensional sphere S_N of radius $N^{1/2}$ centered in the origin and a set of $p = \alpha N$ independent random half spaces. $\Pi_\mu = \{\boldsymbol{J} \in \mathbf{R}^N : N^{-1/2}(\boldsymbol{\xi}^{(\mu)}, \boldsymbol{J}) \geq k\}$ $(\mu = 1, \dots p)$, where $\boldsymbol{\xi}^{(\mu)}$ are i.i.d. random vectors with i.i.d. Bernoulli components $\xi_j^{(\mu)}$ and k is the distance of Π_μ from the origin. The problem is to find the maximum value of α such that the volume of the intersection of S_N with $\cap\Pi_\mu$ *behaves like* $e^{-N\,const}\sigma_N$, *where* σ_N *is the volume of* S_N.

This geometrical question is motivated by the problem of the retrieval of patterns in neural networks. The retrieval of patterns depends on the neural dynamic. The neural dynamic is defined as an evolution of the space of neural activities $\sigma \equiv (\sigma_1, \dots, \sigma_N)$, $\sigma_i = \pm 1$, generated by the threshold dynamics

$$\sigma_i(t+1) = \text{sign}\{\sum_{j=1, j\neq i}^{N} J_{ij}\sigma_j(t)\} \quad (i = 1, \dots, N), \tag{1}$$

where $\sigma(t), \sigma(t+1)$ are the vectors of neural activities at time t and $t+1$ respectively. The value $+1$ for σ_i represents an "active neuron" while -1 is a quiescent neuron. In this language the retrieval means that the activities $\sigma(t)$ converge, for suitable initial conditions, to some pattern $\boldsymbol{\xi}^{(\mu)}$.

The interaction (or synaptic) matrix $\{J_{ij}\}$ (not necessarily symmetric) depends on the concrete model, but usually it satisfies the conditions

$$\sum_{j=1, j\neq i}^{N} J_{ij}^2 = NR \quad (i = 1, \dots, N), \tag{2}$$

where R is some fixed number which could be taken equal to 1.

The problem is to find an interaction matrix $\{J_{ij}\}$ such that some set ξ of chosen vectors $\xi \equiv \{\boldsymbol{\xi}^{(\mu)}\}_{\mu=1}^{p}$ (patterns) are the fixed points of the dynamics (1). This implies the conditions:

$$\xi_i^{(\mu)} \sum_{j=1, j\neq i}^{N} J_{ij}\xi_j^{(\mu)} > 0 \quad (i = 1, \dots, N). \tag{3}$$

Sometimes condition (3) is not sufficient to have $\boldsymbol{\xi}^{(\mu)}$ as the end points of the dynamics. To have some "basin of attraction" (that is some neighbourhood of $\boldsymbol{\xi}^{(\mu)}$, starting from which we for sure arrive in $\boldsymbol{\xi}^{(\mu)}$) one should introduce some positive parameter k and impose the conditions:

$$\xi_i^{(\mu)} \sum_{j=1, j\neq i}^{N} J_{ij}\xi_j^{(\mu)} > k \quad (i = 1, \dots, N). \tag{4}$$

So in this paper we consider only $k > 0$.

Gardner [4] was the first who solved this kind of inverse problem. She asked the question: for which $\alpha = \frac{p}{N}$ the interaction $\{J_{ij}\}$, satisfying (2) and (4) exists? What is the typical fractional volume of these interactions? Since all the conditions (2) and (4) are factorised with respect to i, this problem, after a simple transformation, can be replaced by the following. For the system of $p \sim \alpha N$ i.i.d. random patterns $\xi \equiv \{\boldsymbol{\xi}^{(\mu)}\}_{\mu=1}^{p}$ with i.i.d. $\xi_i^{(\mu)}$ $(i = 1, \dots, N)$ assuming values ± 1 with probability $\frac{1}{2}$, consider

$$\Theta_{N,p}(\xi, k) = \sigma_N^{-1} \int_{(\boldsymbol{J},\boldsymbol{J})=N} d\boldsymbol{J} \prod_{\mu=1}^{p} \theta(N^{-1/2}(\boldsymbol{\xi}^{(\mu)}, \boldsymbol{J}) - k), \tag{5}$$

where the function $\theta(x)$, as usually, is zero in the negative semi-axis and 1 in the positive and σ_N is the Lebesgue measure of N-dimensional sphere of radius $N^{1/2}$. Then, the question of interest is the behaviour of $\frac{1}{N}\log\Theta_{N,p}(\xi,k)$ in the limit $N,p\to\infty$, $\frac{p}{N}\to\alpha$. Gardner [4] had solved this problem by using the so-called replica trick, which is completely non-rigorous from the mathematical point of view but sometimes very useful in the physics of spin glasses (see [7] and references therein). She obtained that for any $\alpha<\alpha_c(k)$, where

$$\alpha_c(k)\equiv(\frac{1}{\sqrt{2\pi}}\int_{-k}^{\infty}(u+k)^2e^{-u^2/2}du)^{-1},$$

the volume of the space of synaptic couplings which have the patterns $\boldsymbol{\xi}^{(\mu)}$ as fixed points decays exponentially with N while for $\alpha\geq\alpha_c$ the volume of the intersection decays as $N\to\infty$ faster than e^{-LN} with any positive L. Our main goal is to prove rigorously the results of [4]. The methods used by us is typical of statistical mechanics of the disordered systems which are systems of N variables *(in our case J_i) with some random function of these variables, which define the "interaction" between them.* Since the randomness of the interaction induces the randomness of *integrals, which appear in the problem,* a natural question arises about the dependence of the integrals from the choice of the random patterns. For this reason we introduce the operation of expectation E with respect to the random patterns. We define, according to statistical mechanics, the free energy of the system $F_{N,p}(\xi,k)$

$$F_{N,p}(\xi,k)=\frac{1}{N}\log\Theta_{N,p}(\xi,k)$$

and the self-averaging of the free-energy as

$$\lim_{N\to\infty}E(F_{N,p}(\xi,k)-EF_{N,p}(\xi,k))^2=0.$$

In other words the free energy is self averaging if, in the limit of large N,it tends in probability to its average E with respect to the patterns.

To formulate our main theorem we should remark that since $\Theta_{N,p}(\xi,k)$ can be zero with nonzero probability (e.g., if for some $\mu\neq\nu$ $\boldsymbol{\xi}^{(\mu)}=-\boldsymbol{\xi}^{(\nu)}$), we cannot, as usually in statistical mechanics, just study

$$\log\Theta_{N,p}(\xi,k).$$

To avoid this difficulty, we take some large enough M and replace the log- function by the function $\log_{(MN)}$, defined as $\log_{(MN)}X=\log\max\left\{X,\ e^{-MN}\right\}$.

Theorem 1 *For any $\alpha \le \alpha_c(k)$ $N^{-1}\log_{(MN)}\Theta_{N,p}(\xi,k)$ is self-averaging in the limit $N,p\to\infty$, $p/N\to\alpha$ and for M large enough there exists*

$$\begin{aligned}&\lim_{N,p\to\infty} E\{N^{-1}\log_{(MN)}\Theta_{N,p}(\xi,k)\}\\ &= \min_{0\le q\le 1}\left[\alpha E\left\{\log \mathrm{H}\left(\frac{u\sqrt{q}+k}{\sqrt{1-q}}\right)\right\}+\frac{1}{2}\frac{q}{1-q}+\frac{1}{2}\log(1-q)\right],\end{aligned} \qquad (6)$$

where $H(x)\equiv\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt$, u is a Gaussian random variable with zero mean variance 1 and $E\{..\}$ is the averaging with respect to u.

For $\alpha>\alpha_c(k)$ $E\{N^{-1}\log_{(MN)}\Theta_{N,p}(\xi,k)\}\to-\infty$, as $N\to\infty$ and then $M\to\infty$.

We remark here that the self-averaging of $N^{-1}\log\Theta_{N,p}(\xi,k)$ was proven also in [17]. In the next two sections we explain the main ideas underlying the proof.

A complete analysis will be presented elsewhere ([13]).

2. Convexity and Decay of Correlations

It can be easily seen that the Gardner problem (5) is very similar to problems of statistical mechanics, where the integrals with respect to N variables in the limit $N\to\infty$ are studied. But due to technical reasons it is not convenient to study directly the model (5) with θ functions. That is why we use a common trick: substitute the θ-functions appearing in the expression of the partition function (5) by some smooth functions which depend on a small parameter ε and tend, as $\varepsilon\to 0$, to the θ-functions. We choose for this purpose $\mathrm{H}(x\varepsilon^{-1/2})$ with $\mathrm{H}(x)$ defined in Theorem 1 but the particular form of this function is not important for us. The most important fact is that its logarithm should be a convex function. To substitute in (5) the integration over S_N by the integration over the whole $\mathbf{R}^N$ we use another well known trick in statistical mechanics. We add to the Hamiltonian a term depending on the additional free parameter z. At the end of our considerations we can choose this parameter in order to provide the condition that for large N only a small neighborhood of S_N gives the main contribution to our integral. Thus, we consider the Hamiltonian of the form

$$\begin{aligned}\mathcal{H}_{N,p}(\boldsymbol{J},\xi,k,h,z,\varepsilon)\equiv -\sum_{\mu=1}^{p}\log\mathrm{H}\left(\frac{k-(\boldsymbol{\xi}^{(\mu)},\boldsymbol{J})N^{-1/2}}{\sqrt{\varepsilon}}\right)\\ +\frac{z}{2}(\boldsymbol{J},\boldsymbol{J})+h(\boldsymbol{h},\boldsymbol{J}).\end{aligned} \qquad (7)$$

Here the last term $h(\boldsymbol{h},\boldsymbol{J})$ is the scalar product of the variables $\boldsymbol{J}$ with some vector $\boldsymbol{h}$ with independent random components introduced for getting the self averaging of the order parameters of the theory (see below)

[6], [8], [9]. The free energy and the Gibbs average for this Hamiltonian are

$$Z_{N,p}(\xi,k,h,z,\varepsilon)=\sigma_N^{-1}\int d\boldsymbol{J}e^{-\mathcal{H}_{N,p}(\boldsymbol{J},\xi,k,h,z,\varepsilon)},$$

$$\langle\ldots\rangle=\int(\ldots)d\boldsymbol{J}\frac{e^{-\mathcal{H}_{N,p}(\boldsymbol{J},\xi,k,h,z,\varepsilon)}}{Z_{N,p}(\xi,k,h,z,\varepsilon)},$$

$$F_{N,p}(\xi,k,h,z,\varepsilon)\equiv\frac{1}{N}\log Z_{N,p}(\xi,k,h,z,\varepsilon).$$

Now we have the typical problem of statistical mechanics which we solve by a method usually called the cavity method. The idea of the cavity method is to choose one variable, e.g. J_N and to try to express $\langle J_N\rangle$ through the Gibbs average of the others J_i, and then, using the symmetry of the Hamiltonian, write the self-consistent equations for the so-called order parameters of the problem $q\equiv\frac{1}{N}\sum\langle J_i\rangle^2$ and $R\equiv\frac{1}{N}\sum\langle J_i^2\rangle$. This procedure allows us to reduce the problem to a finite number of nonlinear equations. The rigorous version of the cavity method was proposed in [8] and developed in [9], [10],[11],[12]. The key problem of the application of the cavity method is the proof of the vanishing of the correlation functions $\langle J_iJ_j\rangle-\langle J_i\rangle\langle J_j\rangle$ as $N\to\infty$. We derived this property from a geometrical statement contained in theorem 2. This result is the analogous of the result of [2], which allows to prove the vanishing of the correlation functions for a large class of models of statistical mechanics which are generated by convex hamiltonians and so also for the model with the hamiltonian $\mathcal{H}_{N,p}(\boldsymbol{J},\xi,k,h,z,\varepsilon)$. Thus we use general definitions in order to show the generality of this theorem.

Let $\{\Phi_N(\boldsymbol{J})\}_{N=1}^{\infty}$ ($\boldsymbol{J}\in\mathbf{R}^N$) be a system of convex functions which have third derivatives bounded in any compact set. Consider also a system of convex domains $\{\Gamma_N\}_{N=1}^{\infty}$ ($\Gamma_N\subset\mathbf{R}^N$) whose boundaries consist of a finite number (may be depending on N) of smooth pieces. Define the Gibbs measure and the free energy, corresponding to $\Phi_N(\boldsymbol{J})$ in Γ_N:

$$\langle\ldots\rangle_{\Phi_N}\equiv\Sigma_N^{-1}\int_{\Gamma_N}d\boldsymbol{J}(\ldots)e^{-\Phi_N(\boldsymbol{J})},\quad\Sigma_N(\Phi_N)\equiv\int_{\Gamma_N}d\boldsymbol{J}e^{-\Phi_N(\boldsymbol{J})}$$

$$f_N(\Phi_N)\equiv\frac{1}{N}\log\Sigma_N(\Phi_N).$$

Denote $\tilde{\Omega}_N(U)\equiv\{\boldsymbol{J}:\Phi_N(\boldsymbol{J})\le U\}$,

$$\Omega_N(U)\equiv\tilde{\Omega}_N(U)\cap\Gamma_N,\tag{8}$$

$\mathcal{D}_N(U) \equiv \tilde{\mathcal{D}}_N(U) \cap \Gamma_N$, where $\tilde{\mathcal{D}}_N(U)$ is the boundary of $\tilde{\Omega}_N(U)$. Define also

$$f_N^*(U) = \frac{1}{N} \log \int_{J \in \mathcal{D}_N(U)} d\boldsymbol{J} e^{-NU}.$$

Theorem 2 *Let the functions $\Phi_N(\boldsymbol{J})$ satisfy the conditions:*

$$\frac{d^2}{dt^2} \Phi_N(\boldsymbol{J} + t\mathbf{e})|_{t=0} \geq C_0 > 0,$$

$$\Phi_N(\boldsymbol{J}) \geq C_1(\boldsymbol{J}, \boldsymbol{J}) - NC_2, \tag{9}$$

$$|\nabla \Phi_N(\boldsymbol{J})| \leq N^{1/2} C_3(U) \ \ (\boldsymbol{J} \in \tilde{\Omega}_N(U)), \tag{10}$$

where $\mathbf{e}$ *is an arbitrarily direction* ($|\mathbf{e}| = 1$), $C_0, C_1, C_2, C_3(U)$ *are some positive N-independent constants and $C_3(U)$ is continuous in U,* ($U > U_{min} \equiv \min_{J \in \Gamma_N} N^{-1} \Phi_N(\boldsymbol{J}) \equiv N^{-1} \Phi_N(\boldsymbol{J}^*)$).
Assume also that there exists some finite N-independent C_4 such that $f_N(\Phi_N) \geq -C_4$.

Then for any $U > U_{min}$

$$f_N^*(U) = \min_{z > 0} \{ f_N(z \Phi_N) + zU \} + O(N^{-1} \log N), \tag{11}$$

and for any $\mathbf{e} \in \mathbf{R}^N$ ($|\mathbf{e}| = 1$) *and any natural p*

$$\langle (\dot{\boldsymbol{J}}, \mathbf{e})^p \rangle_{\Phi_N} \leq C(p), \quad \frac{1}{N^2} \sum_{i,j} \langle \dot{J}_i \dot{J}_j \rangle_{\Phi_N}^2 \leq \frac{C(2)}{N} \quad (\dot{J}_i \equiv J_i - \langle J_i \rangle_{\Phi_N}) \tag{12}$$

with some positive N-independent $C(p)$.

Let us explain the role of convexity in the proof of theorem (2). The Gibbs average of any function of the linear combination $(\boldsymbol{J}, \mathbf{e})$ ($|\mathbf{e}| = 1$) can be expressed in terms of a two-dimensional integral with respect to the energy U (the value of the hamiltonian) and $c = (\boldsymbol{J}, \mathbf{e}) N^{-1/2}$. The additional function, which appears under this change of variables is the "partial entropy", given by the logarithm of the volume of the intersections of the level surfaces of the Hamiltonian with the hyperplanes $(\boldsymbol{J}, \mathbf{e}) = cN^{1/2}$. We study these intersections using a theorem of classical geometry known since the nineteenth century as the Brunn-Minkowski theorem [3]. From this theorem we obtain that the "partial entropy" is a concave function of (U, c). Thus we can apply the Laplace method to evaluate the Gibbs averages. So we obtain the vanishing of the correlation functions, which allows us to find the expression for the free energy.

A similar idea was used in [1] where the results of [2] (also based on the Brunn-Minkowski theorem) have been used. We would like to remark that, differently from [1], we cannot just use the results of [2] because they are true for $\mathbf{R}^N$ while the most nontrivial part of our proof (i.e. the limiting transition $\varepsilon \to 0$) is based on similar results for the intersections of p random half spaces.

We show now more explicitly how to realize the above ideas. For any $U > 0$ consider the set $\Omega_N(U)$ defined in (8). Since $\Phi_N(\boldsymbol{J})$ is a convex function, the set $\Omega_N(U)$ is also convex and $\Omega_N(U) \subset \Omega_N(U')$, if $U < U'$. Let

$$\begin{array}{c} V_N(U) \equiv \mathrm{mes}(\Omega_N(U)), \quad S_N(U) \equiv \mathrm{mes}(\mathcal{D}_N(U)), \\ F_N(U) \equiv \int_{\boldsymbol{J}\in\mathcal{D}_N(U)} |\nabla\Phi_N(\boldsymbol{J})|^{-1} dS_J. \end{array} \tag{13}$$

Here and below the symbol mes(...) means the Lebesgue measure of the correspondent dimension.

Then it is easy to see that the partition function Σ_N can be represented in the form

$$\begin{array}{c} \Sigma_N = N\int_{U>U_{min}} e^{-NU} F_N(U)dU = \int_{U>U_{min}} e^{-NU} \frac{d}{dU} V_N(U)dU \\ = N\int_{U>U_{min}} e^{-NU} V_N(U)dU. \end{array} \tag{14}$$

Here we have used the relation $F_N(U) = N^{-1}\frac{d}{dU}V_N(U)$ and the integration by parts.

Besides, for a chosen direction $\mathbf{e} \in \mathbf{R}^N$ $(|\mathbf{e}| = \mathbf{1})$, and any real c consider the hyper-plane

$$\mathcal{A}(c,\mathbf{e}) = \left\{\boldsymbol{J} \in \mathbf{R}^N : (\boldsymbol{J},\mathbf{e}) = N^{1/2}c\right\}$$

and denote

$$\begin{array}{c} \Omega_N(U,c) \equiv \Omega_N(U) \cap \mathcal{A}(c,\mathbf{e}), \quad V_N(U,c) \equiv \mathrm{mes}(\Omega_N(U,c)), \\ \mathcal{D}_N(U,c) \equiv \mathcal{D}_N(U) \cap \mathcal{A}(c,\mathbf{e}), \quad F_N(U,c) \equiv \int_{\boldsymbol{J}\in\mathcal{D}_N(U,c)} |\nabla\Phi_N(\boldsymbol{J})|^{-1} dS_J. \end{array} \tag{15}$$

Then, since $F_N(U,c) = N^{-1}\frac{\partial}{\partial U}V_N(U,c)$, we obtain

$$\begin{array}{c} \Sigma_N = N\int dcdUe^{-NU}F_N(U,c) = N\int dcdUe^{-NU}V_N(U,c), \\ \langle(\boldsymbol{J},\mathbf{e})^p\rangle_{\Phi_N} = \frac{N^{p/2}\int dcdUc^p e^{-NU}V_N(U,c)}{\int dcdUe^{-NU}V_N(U,c)}. \end{array} \tag{16}$$

Denote

$$s_N(U) \equiv \frac{1}{N}\log V_N(U), \quad s_N(U,c) \equiv \frac{1}{N}\log V_N(U,c). \tag{17}$$

The functions $s_N(U)$ and $s_N(U,c)$ are *the complete* and the "partial" entropies mentioned before and formula (16) is the two dimensional integral on the energy levels U and on the hyper planes $\mathcal{A}(c,\mathbf{e})$. Then the relations (14), (16) give us

$$\begin{array}{c} \Sigma_N = N\int \exp\{N(s_N(U)-U)\}dU, \\ \left\langle (\dot{\boldsymbol{J}},\mathbf{e})^p \right\rangle_{\Phi_N} = N^{p/2}\left\langle (c-\langle c\rangle_{(U,c)})^p \right\rangle_{(U,c)}, \end{array} \tag{18}$$

where

$$\langle ... \rangle_{(U,c)} \equiv \frac{\int dUdc(...)\exp\{N(s_N(U,c)-U)\}}{\int dUdc\exp\{N(s_N(U,c)-U)\}}. \tag{19}$$

The equations (11) and (12) can be obtained by the standard Laplace method, if we prove that $s_N(U)$ and $s_N(U,c)$ are concave functions and they are strictly concave in the neighbourhood of the points U^* and (U^*,c^*) of maximum of the functions $(s_N(U)-U)$ and $(s_N(U,c)-U)$. To prove this we apply the theorem of Brunn-Minkowski from classical geometry (see e.g. [3]) to the functions $s_N(U)$ and $s_N(U,c)$. To formulate this theorem we need some extra definitions.

Consider two bounded sets in $\mathcal{A},\mathcal{B}\subset\mathbf{R}^N$. For any positive α and β

$$\alpha\mathcal{A}\times\beta\mathcal{B}\equiv\{\mathbf{s}:\mathbf{s}=\alpha\mathbf{a}+\beta\mathbf{b},\mathbf{a}\in\mathcal{A},\mathbf{b}\in\mathcal{B}\}.$$

$\alpha\mathcal{A}\times\beta\mathcal{B}$ is the Minkowski sum of $\alpha\mathcal{A}=$ and $\beta\mathcal{B}$.

The one-parameter family of bounded sets $\{\mathcal{A}(t)\}_{t_1^*\le t\le t_2^*}$ is a convex one- parameter family, if for any positive $\alpha<1$ and $t_{1,2}\in[t_1^*,t_2^*]$ they satisfy the condition

$$\mathcal{A}(\alpha t_1+(1-\alpha)t_2)\supset\alpha\mathcal{A}(t_1)\times(1-\alpha)\mathcal{A}(t_2).$$

Theorem of Brunn-Minkowski *Let* $\{\mathcal{A}(t)\}_{t_1^*\le t\le t_2^*}$ *be some convex one-parameter family. Consider* $R(t)\equiv(\mathrm{mes}\mathcal{A}(t))^{1/N}$. *Then* $\frac{d^2R(t)}{dt^2}\le 0$ *and* $\frac{d^2R(t)}{dt^2}\equiv 0$ *for* $t\in[t_1',t_2']$ *if and only if all the sets* $\mathcal{A}(t)$ *for* $t\in[t_1',t_2']$ *are homothetic to each other.*

For the proof of this theorem see, e.g., [3].

To use this theorem for the proof of (11) let us observe that the family $\{\Omega_N(U))\}_{U>U_{min}}$ is a convex one-parameter family and then, according to the Brunn-Minkowski theorem, the function $R(U)=(V_N(U))^{1/N}$ is a concave function. We get that $s_N(U)$ is a concave function:

$$\frac{d^2}{dU^2}s_N(U)=\frac{d^2}{dU^2}\log R(U)=\frac{R''(U)}{R(U)}-\left(\frac{R'(U)}{R(U)}\right)^2\le-\left(\frac{R'(U)}{R(U)}\right)^2.$$

$\frac{R'(U)}{R(U)} = \frac{d}{dU}s_N(U) > 1$ for $U < U^*$, and even if $\frac{d}{dU}s_N(U) = 0$ for $U > U^*$, we obtain that $\frac{d}{dU}(s_N(U) - U) = -1$. Thus, using the standard Laplace method, we get

$$f_N(\Phi_N) = s_N(U^*) - U^* + O(\tfrac{\log N}{N}) = \tfrac{1}{N}\log V_N(U^*) - U^* + O(\tfrac{\log N}{N}),\\ U_* \equiv \tfrac{1}{N}\langle \Phi_N \rangle_{\Phi_N} = U^* + o(1). \tag{20}$$

Using the condition (10), and taking $\boldsymbol{J}^*$, which is the minimum point of $\Phi_N(\boldsymbol{J})$, we get

$$V_N(U^*) \geq N^{-1} \int_{\boldsymbol{J} \in \mathcal{D}_N(U^*)} |(\boldsymbol{J} - \boldsymbol{J}^*, \nabla\Phi_N(\boldsymbol{J}))||\nabla\Phi_N(\boldsymbol{J})|^{-1} dS_{\boldsymbol{J}}\\ \geq S_N(U^*) \frac{U^* - U_{min}}{\max_{\boldsymbol{J} \in \mathcal{D}_N(U^*)} |\nabla\Phi_N(\boldsymbol{J})|} = N^{-1/2} S_N(U^*) C(U^*). \tag{21}$$

On the other hand, for any $U < U^*$

$$\frac{S_N(U)}{N^{1/2} V_N(U)} \geq \min_{\boldsymbol{J} \in \mathcal{D}_N(U)} |\nabla\Phi_N(\boldsymbol{J})| \frac{F_N(U)}{N^{1/2} V_N(U)} =\\ \geq N^{1/2} \min_{\boldsymbol{J} \in \mathcal{D}_N(U)} \frac{U - U_{min}}{|\boldsymbol{J} - \boldsymbol{J}^*|} \frac{d}{dU} s_N(U) \geq \tilde{C} \frac{d}{dU} s_N(U) > \tilde{C}. \tag{22}$$

Here we have used (15) and (9). Thus the same inequality is valid also for $U = U^*$. Inequalities (22) and (21) imply that

$$\frac{1}{N} \log S_N(U^*) = \frac{1}{N} \log V_N(U^*) + O(\frac{\log N}{N}).$$

Combining this relation with (20) we get (11).

Let us observe also that for any (U_0, c_0) and (δ_U, δ_c) the family $\{\Omega_N(U_0 + t\delta_U, c_0 + t\delta_c\}_{t \in [0,1]}$ is a convex one-parameter family and then, according to the Brunn-Minkowski theorem the function $R_N(t) \equiv V^{1/N}(U_0 + t\delta_U, c_0 + t\delta_c)$ is concave. But since in our consideration $N \to \infty$, to obtain that this function is strictly concave in some neighbourhood of the point (U^*, c^*) of maximum of $s_N(U, c) - U$, we shall use some lemma, which is the corollary from the theorem of Brunn-Minkowski.

Lemma *Consider the convex set* $\mathcal{M} \subset \mathbf{R^N}$ *whose boundary consists of a finite number of smooth pieces. Let the convex one-parameter family* $\{\mathcal{A}(t)\}_{t_1^* \leq t \leq t_2^*}$ *be given by the intersections of* $\mathcal{M}$ *with the parallel the hyper-planes* $\mathcal{B}(t) \equiv \{\boldsymbol{J} : (\boldsymbol{J}, \mathbf{e}) = tN^{1/2}\}$. *Suppose that there is some smooth piece* $\mathcal{D}$ *of the boundary of* $\mathcal{M}$, *such that for any* $\boldsymbol{J} \in \mathcal{D}$ *the minimal normal curvature satisfies the inequality* $N^{1/2}\kappa_{min}(\boldsymbol{J}) > K_0$, *and the Lebesgue measure* $S(t)$ *of the intersection* $\mathcal{D} \cap \mathcal{A}(t)$ *satisfies the bound*

$$S(t) \geq N^{1/2} V(t) C(t), \tag{23}$$

where $V(t)$ is the volume of $\mathcal{A}(t)$. Then $\frac{d^2}{dt^2}V^{1/N}(t) \le -K_0 C(t) V^{1/N}(t)$.

As far as we know, the Gardner problem is one of the first problems of spin glass theory completely solved (i.e. for all values of α and k) in a rigorous way. The explanation is that the problem (5) can be reduced to the problem with the convex Hamiltonian (7) in a convex configuration space. It is just this convexity that allows us to prove the vanishing of all correlation functions for all values of α and k, while e.g. in the Hopfield and Sherrington-Kirkpatrick models the vanishing is valid only for small enough α or for high temperatures (see [7] for the physical theory and [11], [12], [14], [15] for the respective rigorous results). Also for the Gardner-Derrida [5] model there is only a justification of the Replica Simmetry solution in a certain region of parameters (see [16]).

3. The Cavity Method and the Limit $\varepsilon \to 0$

As it was mentioned in Section 2, the vanishing of correlation functions is the key problem of applying of the cavity method to the model (7). It allows us to derive the selfconsistent equations for the order parameters of the model (7) and then find the expression for the free energy, which we use for deriving the result of Theorem 1 when $\varepsilon \to 0$.

Theorem 3 *For any $\alpha, k \ge 0$ and $z > 0$ the functions $F_{N,p}(\xi, k, h, z, \varepsilon)$ are self-averaging in the limit $N, p \to \infty$, $\alpha_N \equiv \frac{p}{N} \to \alpha$:*

$$E\left\{(F_{N,p}(\xi,k,h,z,\varepsilon) - EF_{N,p}(\xi,k,h,z,\varepsilon)\})^2\right\} \to 0 \qquad (24)$$

and, if ε is small enough, $\alpha < 2$ and $z \le \varepsilon^{-1/3}$, then there exists

$$\begin{array}{c}\lim_{N,p\to\infty,\alpha_N\to\alpha} E\{F_{N,p}(\xi,k,h,z,\varepsilon)\} = F(\alpha,k,h,z,\varepsilon), \\ F(\alpha,k,h,z,\varepsilon) \equiv \max_{R>0} \min_{0\le q\le R} \left[\alpha E\left\{\log \mathrm{H}\left(\frac{u\sqrt{q}+k}{\sqrt{\varepsilon+R-q}}\right)\right\}\right. \\ \left. +\frac{1}{2}\frac{q}{R-q} + \frac{1}{2}\log(R-q) - \frac{z}{2}R + \frac{h^2}{2}(R-q)\right],\end{array} \qquad (25)$$

where u is a Gaussian random variable with zero mean and variance 1.

Let us note that the bound $\alpha < 2$ is not important for us, because for any $\alpha > \alpha_c(k)$ ($\alpha_c(k) < 2$ for any k) the free energy of the partition function $\Theta_{N,p}(\xi, k)$ tends to $-\infty$, as $N \to \infty$ (see Theorem 1 for the exact statement). The bound $z < \varepsilon^{-1/3}$ also is not a restriction for us. We could need to consider $z > \varepsilon^{-1/3}$ only if, applying the result on $f_N^*(U)$ of theorem 2 to the Hamiltonian $\mathcal{H}_{N,p}(\boldsymbol{J}, \xi, k, h, z, \varepsilon)$, we obtain that the point of minimum $z_{min}(\varepsilon)$ does not satisfy this bound. But it is shown in theorem 1, that for any $\alpha < \alpha_c(k)$ $z_{min}(\varepsilon) < \overline{z}$ with some finite $\overline{z}$ depending only on k and α.

Our last step is the limiting transition $\varepsilon \to 0$, i.e. the proof that θ-functions in (5) can be replaced by $\mathrm{H}(\frac{x}{\sqrt{\varepsilon}})$ with a small difference when ε is small enough. It is the most difficult step from the technical point of view. It is rather straightforward to obtain that the free energy of (7) is an upper bound of $\frac{1}{N}\log\Theta_N, p(\xi,k)$. But the estimate from below is much more complicated. The problem is that to estimate the difference between the free energies corresponding to the two Hamiltonians we, as a rule, need to have them defined in a common configuration space, or at least, we need to know some a priori bounds for some Gibbs averages. In the case of the Gardner problem we do not possess this information. That is why we need to apply our geometrical theorem not only to the model (7) (for these purposes it would be enough to apply the results of [2]) but also to some models, interpolating between (7) and (5), with a complicated random (but convex) configuration space.

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