CHARACTERIZATION OF THE GAUSSIAN DISTRIBUTION ON GROUPS BY THE INDEPENDENCE OF LINEAR STATISTICS

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1. In 1953, Skitovich [1] and Darmois [2] proved the following

THEOREM A. Let $\xi_1, \ldots, \xi_s$ be independent random variables. If the linear forms $L_1 = \alpha_1\xi_1 + \ldots + \alpha_s\xi_s$ and $L_2 = \beta_1\xi_1 + \ldots + \beta_s\xi_s$, where all the coefficients are nonzero, are independent then the random variables are normal.

In this paper we will describe fully the locally compact Abelian groups onto which this characterization theorem is carried over.

Let X be a locally compact separable Abelian metric group (referred to below as a group), $Y = X^*$ be the group of its characters, (x, y) be the value of character $y \in Y$ on the element $x \in X$. A convolution of two distributions μ and ν, the characteristic function of the distribution μ and the distribution $\bar{\mu}$ are defined in the usual manner:

$$(\mu * \nu)(E) = \int_X \mu(E - x)\, d\nu(x), \quad \widehat{\mu}(y) = \int_X (x, y)\, d\mu(x), \quad \bar{\mu}(E) = \mu(-E).$$

Denote by E_x the singular distribution concentrated at the point $x \in X$ and by $D(X)$ the set of singular distributions on the group X. Denote by $I(X)$ the set of translations of the Maar distributions m_K of the compact subgroups K of the group X and by $\sigma(\mu)$ the support of the distribution μ.

If G is a closed subgroup of the group X, we denote by $G^{\perp} = \{y \in Y\colon\ (x, y) = 1 \text{ for all } x \in G\}$ its annihilator, by $\mathbf{R}$, $\mathbf{Z}$, T, $\mathbf{Z}(q)$ the groups of the real numbers, integers, rotations of a circle and roots of unity of the q-th power, respectively. We will utilize in this paper some results of the structure theory of locally compact Abelian groups and Pontryagin's duality theory (cf. [3]).

Definition 1 [4]. A distribution γ on a group X is called Gaussian if its characteristic function can be represented as

$$\widehat{\gamma}(y) = (x, y)\exp\{-\varphi(y)\},$$

where $x \in X$, $\varphi(y)$ is a continuous nonnegative function on Y satisfying the equation

$$\varphi(y_1 + y_2) + \varphi(y_1 - y_2) = 2[\varphi(y_1) + \varphi(y_2)], \quad y_1, y_2 \in Y.$$

Denote by $\Gamma(X)$ the set of Gaussian distributions on the group X. As it is shown in [4] if $\gamma \in \Gamma(X)$, then $\sigma(\gamma)$ is a coset of a certain connected subgroup in X.

Let $n \in \mathbf{Z}$. Consider the mapping $X \to X$ defined by the formula $x \to nx$. The image of the group X under this mapping will be denoted by nX.

The set of integers $\{\alpha_j\}$ will be called admissible for the group X if for all j $\alpha_j X \neq \{0\}$ is fulfilled. Let $\xi_1, \ldots, \xi_s$ be random variables with values in X. The condition of admissibility of the set $\{\alpha_j\}$, when considering the linear form $\alpha_1\xi_1 + \ldots + \alpha_s\xi_s$, is a group analog of the condition $\alpha_j \neq 0$ for all j in the case when $X = \mathbf{R}$.

THEOREM 1. Let $\xi_1, \ldots, \xi_s$ be independent random variables with values in the group X and distributions $\mu_1, \ldots, \mu_s$; $\{\alpha_j\}$ and $\{\beta_j\}$ be sets of integers admissible for X. Assume that the linear forms $L_1 = \alpha_1\xi_1 + \ldots + \alpha_s\xi_s$ and $L_2 = \beta_1\xi_1 + \ldots + \beta_s\xi_s$ are independent. Then

i) if the group X is topologically isomorphic to a group of the form

$$X \approx \mathbf{R}^n + \mathscr{D}, \tag{1}$$

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where $n \geq 0$ and $\mathscr{D}$ is a discrete group which contains no nonzero elements of a finite order then $\mu_j \in \Gamma(X)$, $1 \leq j \leq s$,

ii) if $2X = \{0\}$†, then $\mu_j \in \mathscr{D}(X)$, $1 \leq j \leq s$,

iii) if the group X is topologically isomorphic to the group $X \approx Z(3)$, then either $\mu_j \in \mathscr{D}(X)$, $1 \leq j \leq s$ or $\mu_{j_1} = \mu_{j_2} = m_X$ for at least two distributions μ_{j_1}, μ_{j_2}, the remaining μ_j being arbitrary.

Theorem 1 is exact in the following sense.

THEOREM 2. Let the group X not be isomorphic topologically to the groups stipulated in Theorem 1. Then there exist independent random variables $\xi_1, \ldots, \xi_s$ with values in X and with distributions $\mu_1, \ldots, \mu_s$ and also admissible for X sets of integers $\{\alpha_j\}$ and $\{\beta_j\}$ such that the linear forms $\{\alpha_j\}$ and $\{\beta_j\}$, such that the linear forms $L_1 = \alpha_1\xi_1 + \ldots + \alpha_s\xi_s$ and $L_2 = \beta_1\xi_1 + \ldots + \beta_s\xi_s$ are independent and $\mu_j \notin I(X)$, $1 \leq j \leq s$.

Before proceeding to the theorems, we note that if $\xi_1, \ldots, \xi_s$ are independent, random variables with values in the group X and distributions $\mu_1, \ldots, \mu_s$, then the linear forms $L_1 = \alpha_1\xi_1 + \ldots + \alpha_s\xi_s$ and $L_2 = \beta_1\xi_1 + \ldots + \beta_s\xi_s$, $\alpha_j, \beta_j \in \mathbf{Z}$ are independent if and only if the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\prod_{j=1}^{s} \hat{\mu}_j(\alpha_j u + \beta_j v) = \prod_{j=1}^{s} \hat{\mu}_j(\alpha_j u) \prod_{j=1}^{s} \hat{\mu}_j(\beta_j v), \quad u, v \in Y. \tag{2}$$

The following lemmas will be required in the sequel.

LEMMA 1. Let the group X be such that $Y = X^*$ is a connected compact group; $\xi_1, \ldots, \xi_s$ are independent random variables with values in X and distributions $\mu_1, \ldots, \mu_s$; $\{\alpha_j\}$, $\{\beta_j\}$ are sets of integers different from zero. Then if the linear forms $L_1 = \alpha_1\xi_1 + \ldots + \alpha_s\xi_s$ and $L_2 = \beta_1\xi_1 + \ldots + \beta_s\xi_s$ are independent, it follows that $\mu_j \in \mathscr{D}(X)$, $1 \leq j \leq s$.

Proof. Two cases are possible.

1. $Y \not\approx T$. Then there exists a continuous monomorphism $\psi: \mathbf{R} \to Y$ whose image $\psi(\mathbf{R})$ is dense in Y [3, p. 518]. Consider the contraction of the characteristic function $\hat{\mu}_j(y)$ onto $\psi(\mathbf{R})$. Evidently, $\hat{\mu}_j(\psi(t))$, $t \in \mathbf{R}$ is a characteristic function on $\mathbf{R}$ satisfying equation (2). By Theorem A

$$\hat{\mu}_j(\psi(t)) = \exp\{-a_j t^2 + ib_j t\},$$

where $a_j \geq 0$, $-\infty < b_j < \infty$, $1 \leq j \leq s$.

Let V be an arbitrary neighborhood of zero in Y. Since ψ is a monomorphism and $\overline{\psi(\mathbf{R})} = Y$, one can select a sequence $t_n \to +\infty$, such that $\psi(t_n) \in V$ for all n. If $a_j > 0$ for some j, then $|\hat{\mu}_j(\psi(t_n))| = \exp\{-a_j t_n^2\} \to 0$ as $t_n \to +\infty$ and this contradicts the continuity of $\hat{\mu}_j(y)$. Hence, $a_j = 0$, $1 \leq j \leq s$. Therefore, $|\hat{\mu}_j(\psi(t))| \equiv 1$, $t \in \mathbf{R}$ and since $\psi(\mathbf{R})$ is dense in Y, we have $|\mu_j(y)| \equiv 1$ for $y \in Y$, $1 \leq j \leq s$. Hence $\hat{\mu}_j(y) = (x_j, y)$, i.e., $\mu_j = E_{x_j}$, $x_j \in X$. The lemma is thus proved in the first case.

2. $Y \approx T$. Then $X \approx \mathbf{Z}$. Without loss of generality, it can be assumed that $X = \mathbf{Z}$ and μ_j can be viewed as a distribution on the group with 2π-periodic characteristic functions also satisfying Eq. (2). By Theorem A, $\hat{\mu}_j(t) = \exp\{-a_j t^2 + ib_j t\}$, $a_j \geq 0$, $-\infty < b_j < \infty$. Since the functions $\hat{\mu}_j(t)$ must be 2π-periodical, it follows that $a_j = 0$, $b_j \in \mathbf{Z}$, i.e., $\mu_j \in \mathscr{D}(\mathbf{Z})$, $1 \leq j \leq s$. The lemma is thus proved.

LEMMA 2. Let μ be an arbitrary distribution on the group X, H be a closed subgroup in $Y = X^*$. If the characteristic function $\hat{\mu}(y) = 1$ for all $y \in H$, then $\sigma(\mu) \subset H^{\perp}$.

We omit the proof of this well-known assertion in view of its simplicity.

Proof of Theorem 1. i) It is sufficient to prove the theorem for the group $X = \mathbf{R}^n + \mathscr{D}$. Set

$$\nu_j = \mu_j * \bar{\mu}_j, \quad 1 \leqslant j \leqslant s. \tag{3}$$

†See [3, p. 523] for a description of groups all of whose elements which differ from zero are of order q, where q is a prime number.

It follows from Eq. (2) that it is satisfied — along with the characteristic functions $\hat{\mu}_j(y)$ — also by the characteristic functions $\hat{\nu}_j(y)$, $1 \le j \le s$. Note that $\hat{\nu}_j(y) \ge 0$, $y \in Y$ and consider the contractions of the functions $\hat{\nu}_j(y)$ onto the subgroup $H = \mathscr{D}^*$. Since H is a connected compact group, it follows from Lemma 1 that $\hat{\nu}_j(y) \equiv 1$, $y \in H$. By Lemma 2 $\sigma(\nu_j) \subset H^{\perp} = \mathbf{R}^n$ and then it is easy to derive from Theorem A that $\nu_j \in \Gamma(\mathbf{R}^n)$. It follows from Eq. (3) that the distribution μ_j can be replaced by its displacement $\mu_j' = \mu_j * E_{x_j}$, $x_j \in X$, so that $\nu_j = \mu_j' * \overline{\mu}_j'$ and $\sigma(\mu_j') \subset \mathbf{R}^n$. From here, applying Cramèr's theorem on the decomposition of the Gaussian distribution in $\mathbf{R}^n$ [5, p. 251] we conclude that $\mu_j' \in \Gamma(\mathbf{R}^n)$. Then $\mu_j \in \Gamma(X)$, $1 \le j \le s$ which proves the theorem in case i).

ii) Because $2X = \{0\}$, the admissible sets for X will be the sets consisting of the odd numbers. Since also $2Y = \{0\}$, Eq. (2) becomes

$$\prod_{j=1}^{s} \widehat{\mu}_j(u+v) = \prod_{j=1}^{s} \widehat{\mu}_j(u) \prod_{j=1}^{s} \widehat{\mu}_j(v), \quad u,\ v \in Y. \tag{4}$$

Setting in (4) $u = v = y$, we obtain that $|\hat{\mu}_j(y)| \equiv 1$ on Y. Hence, $\hat{\mu}_j(y) = (x_j, y)$, i.e., $\mu_j = E_{x_j}$, $1 \le j \le s$. For the case ii) the theorem is also proved.

iii) We can assume without loss of generality that $X = \mathbf{Z}(3)$. Since the admissible sets for the group $\mathbf{Z}(3)$ consist of integers not divisible by 3, it is easy to see that one can assume that $L_1 = \xi_1, \ldots, \xi_s$ and $L_2 = \xi_1 + \ldots + \xi_\ell - \xi_{\ell+1} - \ldots - \xi_s$. Equation (2) becomes

$$\prod_{j=1}^{l} \widehat{\mu}_j(u+v) \prod_{j=l+1}^{s} \widehat{\mu}_j(u-v) = \prod_{j=1}^{s} \widehat{\mu}_j(u) \prod_{j=1}^{l} \widehat{\mu}_j(v) \prod_{j=l+1}^{s} \widehat{\mu}_j(-v). \tag{5}$$

Set $\eta_1 = \xi_1 + \ldots + \xi_\ell$, $\eta_2 = \xi_{\ell+1} + \ldots + \xi_s$. The distributions of the random variables are η_1, η_2. The characteristic functions of the distributions $\nu_1 = \mathop{*}\limits_{j=1}^{l} \mu_j$, $\nu_2 = \mathop{*}\limits_{j=l+1}^{s} \mu_j$ evidently satisfy the equation

$$\widehat{\nu}_1(u+v)\widehat{\nu}_2(u-v) = \widehat{\nu}_1(u)\widehat{\nu}_2(u)\widehat{\nu}_1(v)\widehat{\nu}_2(-v). \tag{6}$$

As it was proved in [6] it follows from (6) that $\nu_1, \nu_2 \in I(\mathbf{Z}(3))$. Therefore, if $\nu_1 \in \mathscr{D}(\mathbf{Z}(3))$, Eq. (6) implies that $\nu_2 \in \mathscr{D}(\mathbf{Z}(3))$ also and then we also have that $\mu_j \in \mathscr{D}(\mathbf{Z} \times (3))$, $1 \le j \le s$. If, however, $\nu_1 = m_{\mathbf{Z}(3)}$, it then follows from Eq. (6) that $\nu_2 = m_{\mathbf{Z}(3)}$ also. Now note that if $\gamma_1 * \gamma_2 = m_{\mathbf{Z}(3)}$ on the group $X = \mathbf{Z}(3)$ then at least one of the distributions $\gamma_j = m_{\mathbf{Z}(3)}$. It is then easy to conclude from (5) that the remaining distributions μ_j can be arbitrary. Theorem 1 is thus proved.

The following lemmas are required for the proof of Theorem 2.

<u>LEMMA 3.</u> Let K be a compact group such that $2K = K$, $K \not\approx \mathbf{Z}(3)$. Then there exist independent random variables ξ_j with values in K and distributions μ_j, $1 \le j \le 4$, such that the linear forms $L_1 = \xi_1 + \xi_2 + \xi_3 + \xi_4$ and $L_2 = \xi_1 + \xi_2 - \xi_3 - \xi_4$ will be independent and $\mu_j \notin I(K) * \Gamma(K)$, $1 \le j \le 4$.

<u>Proof.</u> Denote $f(y) = \hat{m}_K(y)$, $y \in Y = K^*$. It is easy to verify that the function $f(y)$ satisfies the equation

$$f(u+v)f(u-v) = f^2(u)\,|f(v)|^2, \quad u,\ v \in Y.$$

Choose $y_1, y_2 \in Y$ such that the elements $\{y_1, y_2, -y_1, -y_2\}$ will be pairwise distinct and consider on the group K the functions

$$\rho_j(x) = 1 + (1/2)\,[(x, y_j) + \overline{(x, y_j)}], \quad j = 1, 2.$$

Evidently, $\rho_j(x) \ge 0$. Denote by μ_j the distribution on the group K with the density $\rho_j(x)$ (with respect to m_K). It is easy to see that $\mu_1 * \mu_2 = m_K$. Characteristic functions $\hat{\mu}_1(y)$ and $\hat{\mu}_2(y)$ satisfy the equation

$$\mu_1(u+v)\widehat{\mu}_2(u+v)\widehat{\mu}_1(u-v)\widehat{\mu}_2(u-v) = \widehat{\mu}_1^2(u)\widehat{\mu}_2^2(u)\,|\widehat{\mu}_1(v)|^2|\widehat{\mu}_2(v)|^2, \tag{7}$$

$u, v \in Y$. Let ξ_j be independent random variables with the values in the group K possessing the distributions μ_j, $1 \le j \le 4$, where $\mu_1 = \mu_3$, $\mu_2 = \mu_4$. It follows from Eq. (7) that the linear forms $L_1 = \xi_1 + \xi_2 + \xi_3 + \xi_4$ and $L_2 = \xi_1 + \xi_2 - \xi_3 - \xi_4$ are independent. It is evident that $\mu_j \notin I(K) * \Gamma(K)$. The lemma is proved.

Let q be a prime number. Denote by $\mathbf{Z}(q^\infty)$ the multiplicative group (in the discrete topology) of the roots of unity whose powers are powers of the number q. Denote by Δ_q the group of characters of the group $\mathbf{Z}(q^\infty)$ (see, e.g., [3, Sec. 25] for more details about the group Δ_q).

LEMMA 4. Let $X = \Delta_q$. Then there exist independent random variables ξ_1, ξ_2 with the values in Δ_q and with the distributions μ_1, μ_2, such that the linear forms $L_1 = q\xi_1 - \xi_2$ and $L_2 = \xi_1 + q\xi_2$ are independent. It is evident that $\mu_1, \mu_2 \notin I(\Delta_q)$.

Proof. Imbed the group $\mathbf{Z}(q)$ into $\mathbf{Z}(q^\infty)$ and consider the functions

$$f_j(y) = \begin{cases} \widehat{\nu}_j(y), & y \in \mathbf{Z}(q), \\ 0, & y \notin \mathbf{Z}(q), \quad j = 1,2, \end{cases}$$

on the group $\mathbf{Z}(q^\infty)$ where $\hat{\nu}_j(y)$ are arbitrary characteristic functions on the group $\mathbf{Z}(q)$. Functions $f_j(y)$ are continuous, positive definite [7, p. 330] and by the Bochner–Khinchin theorem are characteristic functions of certain distributions μ_j on the group Δ_q. Let ξ_j be a random variable with values in Δ_q possessing the distribution μ_j. To verify that the linear forms $L_1 = q\xi_1 - \xi_2$ and $L_2 = \xi_1 + q\xi_2$ are independent, it is sufficient to verify that the functions $f_1(y)$ and $f_2(y)$ satisfy Eq. (2) which becomes

$$f_1(qu+v) f_2(-u+qv) = f_1(qu) f_2(-u) f_1(v) f_2(qv), \quad u, v \in \mathbf{Z}(q^\infty). \tag{8}$$

If $u, v \in \mathbf{Z}(q)$, relation (8) is evidently fulfilled. If either $u \in \mathbf{Z}(q)$, $v \notin \mathbf{Z}(q)$ or $u \notin \mathbf{Z}(q)$, $v \in \mathbf{Z}(q)$, then the left-hand and the right-hand sides in Eq. (8) vanish. Let $u, v \notin \mathbf{Z}(q)$. Then the right-hand side in Eq. (8) vanishes. If $qu + v$, $-u + qv \in \mathbf{Z}(q)$, then $(q^2 + 1)u \in \mathbf{Z}(q)$ and hence $u \in \mathbf{Z}(q)$ which contradicts the assumption. Thus, either $qu + v \notin \mathbf{Z}(q)$ or $-u + qv \notin \mathbf{Z}(q)$ and the left-hand side of Eq. (8) also vanishes. This proves equality (8) and hence the independence of the linear forms L_1 and L_2 is established. It is also evident that if $\nu_1, \nu_2 \notin I(\mathbf{Z}(q))$, then $\mu_1, \mu_2 \notin I(\Delta_q)$. The lemma is thus proved.

LEMMA 5. Let G be a closed subgroup of the group X, μ be a distribution on G. If $\mu \notin I(G) * \Gamma(G)$, then $\mu \notin I(X) * \Gamma(X)$.

The proof of this lemma is self-evident and is thus omitted.

Proof of Theorem 2. Assume that the group X is not topologically isomorphic to the groups stipulated in Theorem 1. The following cases are possible.

1. The group X contains a subgroup $G \approx \mathbf{Z}(2)$. Let ξ_1 and ξ_2 be independent random variables with values in G and the distributions $\mu_1, \mu_2 \notin I(G)$. Since the group is discrete we have $\Gamma(G) = \mathcal{D}(G)$ and hence $I(G) * \Gamma(G) = I(G)$. From Lemma 5 $\mu_1, \mu_2 \notin I(X) * \Gamma(X)$. As it is easy to see, the linear forms $L_1 = 2\xi_1 - \xi_2$ and $L_2 = \xi_1 + 2\xi_2$ are independent and the sets $\{2, -1\}$, $\{1, 2\}$ are admissible for the group X since by the condition $2X \neq \{0\}$.

2. The group X contains the subgroup $G \approx \mathbf{Z}(3)$. If $3X \neq \{0\}$, let ξ_1 and ξ_2 be independent random variables with values in G possessing the distributions $\mu_1, \mu_2 \notin I(G)$. Since the group G is discrete, it follows that $\Gamma(G)\ \mathcal{D}(G)$; therefore $I(G) * \Gamma(G) = I(G)$. Lemma 5 implies that $\mu_1, \mu_2 \notin I(X) * \Gamma(X)$. The linear forms $L_1 = 3\xi_1 - \xi_2$ and $L_2 = \xi_1 + 3\xi_2$ are evidently independent and the sets $\{3, -1\}$, $\{1, 3\}$ are admissible for the group X.

If $3X = \{0\}$, then since $X \not\approx \mathbf{Z}(3)$, the group X contains the subgroup $K \approx \mathbf{Z}(3) + \mathbf{Z}(3)$. The existence of the required random variables ξ_j and the linear forms L_1, L_2 is now assured by the Lemmas 3 and 5.

3. Group X contains the subgroup $K \approx \mathbf{Z}(q)$ for some prime $q > j$. The existence of the required random variables ξ_j and the linear forms L_1, L_2 also follows from Lemmas 3 and 5.

4. Group X contains no nonzero elements of a finite order. From the structure theorem for locally compact Abelian groups, each group X is topologically isomorphic to the group of the form $X \approx \mathbf{R}^n + G$, where $n \geq 0$ and the group G contains a compact open subgroup K (cf. [3, p. 493]). Since the group is not topologically isomorphic to the group of the form (1) and contains no nonzero elements of a finite order, it follows that $K \neq \{0\}$. However, the compact group K containing no nonzero elements of a finite order is topologically isomorphic to the group of the form

$$K \approx (\Sigma_a)^{\mathfrak{N}} + \bigoplus_{q \in \mathscr{P}} \Delta_q^{\mathfrak{N}_q}, \tag{9}$$

where Σ_a is the group of characters of the group of rational numbers $\mathbf{Q}$ in the discrete topology, $\mathscr{P}$ is the set of all the prime numbers, $\mathfrak{N}, \mathfrak{N}_q$ are cardinal numbers [3, p. 514]. We now observe that for any prime q the group Σ contains the subgroup $G_1 \approx \Delta_q$. Since the group Δ_q is completely disconnected, $\Gamma(\Delta_S) = \mathscr{D}(\Delta_S)$ and hence $I(\Delta_S) * \Gamma(\Delta_q) = I(\Delta_q)$. The existence of the required random variables ξ_j and the linear forms L_1, L_2 now follows from Lemmas 4 and 5. The proof of Theorem 2 is thus completed.

2. Now let $\xi_1, \ldots, \xi_s$ be independent random variables with values in the group X and the distributions $\mu_1, \ldots, \mu_s$; $\{\alpha_j\}$, $\{\beta_j\}$ will be admissible sets of integers for the group X. As we have seen in the proofs of Theorems 1 and 2 the independence of the linear forms $L_1 = \alpha_1\xi_1 + \ldots + \alpha_s\xi_s$ and $L_2 = \beta_1\xi_1 + \ldots + \beta_s\xi_s$ (unlike the situation when $X = \mathbf{R}$) in general does not imply that the characteristic function $\hat{\mu}_j(y)$† does not vanish. In Theorem 3 presented below, we provide a complete description of the groups X for which the independence of the linear forms L_1 and L_2 under the additional condition $\hat{\mu}_j(y) \neq 0$ for all $y \in Y$, $1 \leq j \leq s$ implies that $\mu_j \in \Gamma(X)$, $1 \leq j \leq s$.

<u>THEOREM 3.</u> Let $\xi_1, \ldots, \xi_s$ be independent random variables with the values in group X and distributions μ_j such that $\prod_{j=1}^{s} \hat{\mu}_j(y) \neq 0$ for all $y \in Y$; $\{\alpha_j\}$ and $\{\beta_j\}$ will be admissible for X sets of integers. In order that the independence of linear forms $L_1 = \alpha_1\xi_1 + \ldots + \alpha_s\xi_s$ and $L_2 = \beta_1\xi_1 + \ldots + \beta_s\xi_s$ imply that $\mu_j \in \Gamma(X)$, $1 \leq j \leq s$ it is necessary and sufficient that the group X either contains no nonzero elements of a finite order or $qX = \{0\}$, where q is a prime.‡

To prove the theorem the following lemmas will be required.

<u>LEMMA 6.</u> Let K be a compact group $\alpha, \beta \in \mathbf{Z}$, $\alpha K = K$, f(g) be a continuous nonnegative function on K. Then the inequality

$$\int_{K^2} f(\alpha u + \beta v)\, dm_{K^2}(u, v) \leqslant \int_{K^2} f(\alpha u)\, dm_{K^2}(u, v) + \int_{K^2} f(\beta v)\, dm_{K^2}(u, v), \tag{10}$$

is fulfilled with the equality valid if and only if $f(g) = 0$ for $g \in \beta K$.

<u>Proof.</u> Since $\alpha K = K$, it is easy to verify that

$$\int_{K^2} f(\alpha u + \beta v)\, dm_{K^2}(u, v) = \int_K f(g)\, dm_K(g), \tag{11}$$

$$\int_{K^2} f(\alpha u))\, dm_{K^2}(u, v) = \int_K f(g)\, dm_K(g). \tag{12}$$

Since $f(g) \geq 0$ on K, inequality (10) follows from (11) and (12). Moreover, if this inequality becomes an equality then $f(\beta v) \equiv 0$ on K, i.e., $f(g) = 0$ for $g \in \beta K$.

<u>LEMMA 7.</u> Let $X = \mathbf{Z}(q^\infty)$, $\xi_1, \ldots, \xi_s$ be independent random variables with the values in $\mathbf{Z}(q^\infty)$ and distributions μ_j such that $\hat{\mu}_j(y) > 0$ for all $y \in \Delta_q$, $1 \leq j \leq s$, and $\prod_{j=1}^{s} \hat{\mu}_j \times$

†Proofs of Theorems 1 and 2 imply that the group for which the independence of the linear forms L_1 and L_2 imply that the characteristic functions $\hat{\mu}_j(y)$ do not vanish are either groups of the form (1) or groups for which $2X = \{0\}$.
‡Since the group X such that $qX = \{0\}$ is completely disconnected, it satisfies $\Gamma(X) = \mathscr{D}(X)$.

$(y) = 1$ only if $y = 0$; $\{\alpha_j\}$, $\{\beta_j\}$ be sets of integers different from zero. Then the linear forms $L_1 = \alpha_1\xi_1 + \ldots + \alpha_s\xi_s$ and $L_2 = \beta_1\xi_1 + \ldots + \beta_s\xi_s$ cannot be independent.

Proof. Replacing, if necessary, the random variables ξ_j by the random variables $\xi_j' = d_j\xi_j$ we can assume that the numbers α_j and β_j are mutually prime and the condition $\prod_{j=1}^{s} \times \hat{\mu}_j(y) = 1$ is not violated only for $y = 0$. Indeed, let the equality $\hat{\mu}_j(d_j y) = 1$, $1 \le j \le s$ be fulfilled for some $y \in \Delta_q$. Then $\hat{\mu}_j(d_1, \ldots, d_s y) = 1$, $1 \le j \le s$. Hence, $d_1, \ldots, d_s y = 0$ by the conditions of the lemma and since the group Δ_q contains no nonzero elements of a finite order, it follows that $y = 0$.

Set $f_j(y) = -\ln \hat{\mu}_j(y)$. We then obtain from Eq. (2)

$$\sum_{j=1}^{s} f_j(\alpha_j u + \beta_j v) = \sum_{j=1}^{s} f_j(\alpha_j u) + \sum_{j=1}^{s} f_j(\beta_j v).$$

Integrating this equality over the group Δ_q^2 and interchanging the orders of integration and summation we have

$$\sum_{j=1}^{s} \int_{\Delta_q^2} f_j(\alpha_j u + \beta_j v)\, dm_{\Delta_q^2}(u, v) = \sum_{j=1}^{s} \int_{\Delta_q^2} f_j(\alpha_j u)\, dm_{\Delta_q^2}(u, v) + \sum_{j=1}^{s} \int_{\Delta_q^2} f_j(\beta_j v)\, dm_{\Delta_q^2}(u, v). \tag{13}$$

Since the numbers α_j and β_j are mutually prime, at least one of these numbers is not divisible by q. Therefore either $\alpha_j\Delta_q = \Delta_q$ or $\beta_j\Delta_q = \Delta_q$. Lemma 6 implies from Eq. (13) that in inequality (10) for $f(y) = f_j(y)$, $1 \le j \le s$ equality is valid. Hence $\prod_{j=1}^{s} \hat{\mu}_j(y) = 1$ for $y \in \left(\prod_{j=1}^{s} \alpha_j\beta_i\right) \Delta_q \ne \{0\}$ which contradicts the condition.

LEMMA 8. Let $X = \mathbf{Z}(q)$, where q is a prime number, $\xi_1, \ldots, \xi_s$ will be independent random variables with values in $\mathbf{Z}(q)$ and distributions μ_j such that $\hat{\mu}_j(y) > 0$ for all $y \in \mathbf{Z}(q)$, $1 \le j \le s$ and $\prod_{j=1}^{s} \hat{\mu}_j(y) = 1$ for only $y = 0$, $\{\alpha_j\}$, $\{\beta_j\}$ will be sets of integers none of which is divisible by q. Then the linear forms $L_1 = \alpha_1\xi_1 + \ldots + \alpha_s\xi_s$ and $L_2 = \beta_1\xi_1 + \ldots + \beta_s\xi_s$ cannot be independent.

The proof is completely analogous to the proof of Lemma 7 and is this omitted.

LEMMA 9 [8]. Let μ, μ_1, μ_2 be distributions on the group X. In order that $\mu \in \Gamma(X)$, $\mu = \mu_1 * \mu_2$ imply the inclusion $\mu_j \in \Gamma(X)$, $j = 1, 2$, it is necessary and sufficient that the group X contain no subgroup topologically isomorphic to T.

An element $x_0 \in X$, $x_0 \ne 0$ will be called infinitely divisible if the equation $nx = x_0$ possesses a solution in X for an arbitrary large positive integer n.

LEMMA 10 [9]. Let the group X possess the property: any quotient group of the group $Y = X^*$ possesses an infinitely divisible element. Then if the characteristic function $\hat{\gamma}(y)$ of a distribution γ on X satisfies the conditions: 1) $\hat{\gamma}(y) > 0$ for all $y \in Y$; 2) $\hat{\gamma}(ny) = (\hat{\gamma}(y))^{n^2}$, $n = 2, 3, \ldots$, then $\gamma \in \Gamma(X)$.

Proof of Theorem 3. Necessity. Assume that the group X contains an element x_0 of order q where q is a prime number but $qX \ne \{0\}$. Let G be a subgroup in X generated by the element x_0 and ξ_1, ξ_2 be independent random variables with values in G possessing nondegenerate distributions μ_1, μ_2 such that $\hat{\mu}_1(y)\hat{\mu}_2(y) \ne 0$ for all $y \in G^*$. As it is easily seen, the linear forms $L_1 = q\xi_1 - \xi_2$ and $L_2 = \xi_1 + q\xi_2$ are independent and the sets $\{q - 1\}$ and $\{1, q\}$ are admissible for the group X. Evidently, $\mu_j \notin \Gamma(X)$, $j = 1, 2$.

Sufficiency. Assume that the group X possesses no nonzero elements of a finite order. It is necessary to prove that if the characteristic functions of distributions μ_j satisfy Eq. (2) then $\mu_j \in \Gamma(X)$, $1 \le j \le s$. Evidently, the characteristic functions of distributions $\nu_j = \mu_j * \mu_j$ also satisfy Eq. (2) but here $\hat{\nu}_j(y) > 0$ for all $y \in Y$. Since the group X contains no subgroup topologically isomorphic to T, in view of Lemma 9 it is sufficient to verify that $\nu_j \in \Gamma(X)$, $1 \le j \le s$. We can thus assume from the very beginning that $\hat{\mu}_j \times (y) > 0$ for all $y \in Y$, $1 \le j \le s$. In view of Lemma 2 one can also assume that $\prod_{j=1}^{s} \hat{\mu}_j(y) =$

1 only for $y = 0$. We first verify that in this case $X \approx \mathbf{R}^n + (\Sigma_a)^{\mathfrak{R}}$, where $n > 0$ and is a cardinal number.

By the structure theorem for locally compact Abelian groups, the group X is topologically isomorphic to the group of the form $X \approx \mathbf{R}^n + G$, where $n \geq 0$ and the group G contains a compact open subgroup K. Let $Y \approx \mathbf{R}^n + H$, $H = G^*$. Since C_H is a connected compact group, Lemma 1 implies that $C_H = \{0\}$, i.e., the group H is completely disconnected. Since the group K contains no nonzero elements of a finite order, K is topologically isomorphic to a group of the form (9) and then $C_X \approx \mathbf{R}^n + (\Sigma_a)^{\mathfrak{R}}$. Since the group X contains no nonzero elements of a finite order, C_X is a direct summand in X, i.e., $X = C_X + A$ [3, p. 529]. The group A is now completely disconnnected. Denote $A^* = B$. Since the group A is completely disconnected, group B consists of compact elements [3, p. 496] and since the group B is topologically isomorphic to a subgroup in H, group B is completely disconnected because H is such a group.

Consider an element $b \in B$ and let M_b be a closed subgroup generated by b. Properties of the group B imply that M_b is a zero-dimensional compact monotonic group. Therefore, (cf. [3, p. 517]) the group M_b is topologically isomorphic to the direct sum

$$M_b \approx \bigoplus_{q \in \mathscr{P}} F_q, \tag{14}$$

where $\mathscr{P}$ is the set of prime numbers and each one of the groups F_q is either $\{0\}$ or $\mathbf{Z}(q^{r_q})$ (r_q is a positive integer) of Δ_q. In view of Lemma 7, the case when $F_q = \Delta_q$ for at least one q is impossible. In view of Lemma 8, the set of q such that the subgroup $F_q = \mathbf{Z}(q^{r_q})$ can occur in the decomposition (14) for all possible elements $b \in B$ is finite and consists of divisors of certain numbers from the set $\{\alpha_j, \beta_j\}_{j=1}^{s}$. If the exponents r_q are jointly bounded, then for some $n \in \mathbf{Z}$ the relation $nB = \{0\}$ is valid which implies that $nA = \{0\}$ is also valid. This is, however, impossible since the subgroup $A \subset X$ contains no nonzero elements of a finite order. Hence the group B contains subgroups topologically isomorphic to $\mathbf{Z}(q^{r_q})$ with an arbitrary large r_q for at least one q. We fix such a q. Let $\alpha_j = q^{\ell_j} a_j$, $\beta_j = q^{m_j} \cdot b_j$, $n_j = \min\{\ell_j, m_j\}$, $\ell_j, m_j \geq 0$. Choose $n > \max\limits_{1 \leq j \leq s} \{\ell_j, m_j\}$, such that the group B possesses a subgroup F topologically isomorphic to $\mathbf{Z}(q^n)$. Set $f_j(y) = -\ln \times \hat{\mu}_j(y)$ and note that by the condition $\sum\limits_{j=1}^{s} f_j(y) = 0$ for $y = 0$ only. Taking logarithms on both sides of Eq. (2) we integrate the equality obtained over the group F^2 with respect to the measure dm_{F^2} interchanging the order of summation and integration. We have

$$\sum_{j=1}^{s} \frac{1}{q^{2n}} \sum_{(u,v)\in F^2} f_j(\alpha_j u + \beta_j v) = \sum_{j=1}^{s} \frac{1}{q^{2n}} \sum_{(u,v)\in F^2} f_j(\alpha_j u) + \sum_{j=1}^{s} \frac{1}{q^{2n}} \sum_{(u,v)\in F^2} f_j(\beta_j v). \tag{15}$$

It is easy to verify that

$$\sum_{(u,v)\in F^2} f_j(\alpha_j u + \beta_j v) = q^{n+n_j} \sum_{y \in q^{n_j} F} f_j(y),$$
$$\sum_{(u,v)\in F^2} f_j(\alpha_j u) = q^{n+l_j} \sum_{y \in q^{l_j} F} f_j(y),$$
$$\sum_{(u,v)\in F^2} f_j(\beta_j v) = q^{n+m_j} \sum_{y \in q^{m_j} F} f_j(y).$$

Note also that in view of the choice of number n, relations $q^{\ell_j} \cdot F \neq \{0\}$, $q^{m_j} \cdot F \neq \{0\}$, $1 \leq j \leq s$.

It follows from the above that equality (15) can be rewritten in the form

$$\sum_{j=1}^{s} \frac{1}{q^{n-n_j}} \sum_{y \in q^{n_j} F} f_j(y) = \sum_{j=1}^{s} \frac{1}{q^{n-l_j}} \sum_{y \in q^{l_j} F} f_j(y) + \sum_{j=1}^{s} \frac{1}{q^{n-m_j}} \sum_{y \in q^{m_j} F} f_j(y). \tag{16}$$

Since either $n_j = \ell_j$ or $n_j = m_j$, equality (16) is impossible. We have thus shown that $B = \{0\}$ and hence also $A = \{0\}$, i.e., $X \approx \mathbf{R}^n + (\Sigma_a)^{\mathfrak{R}}$. The problem is thus reduced to a proof of the theorem for a group X of the form indicated above.

Observe that $Y \approx \mathbf{R}^n + Q^{\mathfrak{R}*}$. Since for any integer m, X and Y are groups with single-valued division by m, utilizing Lemma 9 we can, as is the case in the classical situation, assume, without loss of generality, that $\alpha_j = 1$, $1 \leq j \leq s$ and all β_j are rational and distinct.

Fix an arbitrary element $y_0 \in Y$ and take the subgroup $L(y_0) = \{y \in Y: \; y = (m/n)y_0, m, n \in \mathbf{Z}, \; n \neq 0\}$. Evidently, $L(y_0) \approx \mathbf{Q}$. Set $\psi_j(m/n) = f_j((m.n)y_0)$ and consider the contraction of Eq. (2) onto $L(y_0)$. Then the functions $\psi_j(r)$, $r \in \mathbf{Q}$, $1 \leq j \leq s$ satisfy the equation

$$\sum_{j=1}^{s} \psi_j(u + \beta_j v) = A(u) + B(v), \quad u, v \in \mathbf{Q}, \tag{17}$$

where $A(u) = \sum_{j=1}^{s} \psi_j(u)$, $B(v) = \sum_{j=1}^{s} \psi_j(\beta_j v)$. Further arguments, up to the notation, coincide with the proof of the Skitovich–Darmois theorem by the method of finite differences (see, e.g. [10, pp. 185, 186]). All the increments of the arguments should be considered to be rational here. We will obtain that each function $\psi_j(u)$ satisfies the equation

$$\Delta_h^{s+1} \psi_j(u) = 0, \quad u \in \mathbf{Q}, \tag{18}$$

where h is an arbitrary rational number and Δ_h is a finite difference operator $\Delta_h f(u) = f(u + h) - f(u)$.

[For completeness we present the corresponding arguments for the case s = 2. Equation (17) becomes

$$\psi_1(u+\beta_1 v)+\psi_2(u+\beta_2 v)=A(u)+B(v). \tag{19}$$

We increment u and v by the amounts h, $k \in \mathbf{Q}$ such that $h + \beta_2 k = 0$. Since $\beta_1 \neq \beta_2$, it follows that $\ell = h + \beta_1 k \neq 0$. Then

$$\psi_1(u+\beta_1 v+l)+\psi_2(u+\beta_2 v)=A(u+h)+B(v+k). \tag{20}$$

Subtracting Eq. (19) from Eq. (20) we have

$$\psi_1(u+\beta_1 v+l)-\psi_1(u+\beta_1 v)=\Delta_h A(u)+\Delta_k B(v). \tag{21}$$

Setting v = 0 here we obtain

$$\psi_1(u+l)-\psi_1(u)=\Delta_h A(u)+\Delta_k B(0). \tag{22}$$

Subtracting Eq. (22) from Eq. (21) and setting $v = \ell/\beta_1$ we arrive at

$$\psi_1(u+2l)-2\psi_1(u+l)+\psi_1(u)=\Delta_k B(l/\beta_1)-\Delta_k B(0),$$

i.e., $\Delta_\ell{}^2\psi_1(u) = \varepsilon(\ell)$ and hence $\Delta_\ell{}^2\psi_1(u) = 0$. We observe now that $\ell = h(\beta_2 - \beta_1)/\beta_2$ is an arbitrary rational number since h is arbitrary.]

We rewrite Eq. (18) in the form

$$\sum_{k=0}^{s+1} (-1)^{s-k+1} C_{s+1}^{k} \psi_j(u + kh). \tag{23}$$

Fix $h \in \mathbf{Q}$ and denote $L_h = \{\ell h\}_{\ell=-\infty}^{\infty}$ as the subgroup generated by h. Consider Eq. (23) on L_h and note that if $\psi_j(u) = 0$ for any $u \in \mathbf{Z} \cap L_h$, then $\psi_j(u) \equiv 0$ on L_h. Indeed, any solution of Eq. (23) on L_h is of the form $\psi_j(u) = c_0 + c_1 u + \ldots + c_s u^s$, where c_i are constants [11, p. 322]. Since the set $\mathbf{Z} \cap L_h$ is infinite $c_0 = \ldots = c_s = 0$.

Now note that any polynomial of degree not higher than s satisfies Eq. (23). We choose the polynomial $\varphi_j(u) = a_0 + a_1 u + \ldots + a_s u^s$ such that $\psi_j(p) = \varphi_j(p)$ for p = 0, 1, ..., s. Set $\delta_j(u) = \psi_j(u) - \varphi_j(u)$. The function $\delta_j(u)$ also satisfies Eq. (23) for any $h \in \mathbf{Q}$ and conditions $\delta_j(p) = 0$, p = 0, 1, ..., s. Therefore setting in Eq. (23) h = 1, u = 0, ±1, ... we successively obtain that $\delta_j(u) = 0$ for any $u \in \mathbf{Z}$. As it was mentioned above, this implies that $\delta_j(u) = 0$ for any $u \in L_h$ where h is an arbitrary rational number, i.e., $\delta_j \times (u) \equiv 0$ on $\mathbf{Q}$. Thus the function $\psi_j(u)$ is a polynomial of degree at most s on $\mathbf{Q}$. Therefore the functions $\ell_j(u) = \exp\{\psi_j(u)\}$ are continuous on the group G in the topology induced on $\mathbf{Q}$ from $\mathbf{Q}$ and hence are extended up to positive definite functions $\ell_j(u)$ on the group $\mathbf{R}$ satisfying Eq. (2). By Theorem A, $\ell_j(u) = \exp\{-a_j u^2\}$, $u \in \mathbf{R}$. In particular, for any integer n the equality $\ell_j(nu) = (\ell_j(u))^{n^2}$ is valid. Setting in it u = 1 we obtain

$\hat{\mu}_j(ny_0) = (\hat{\mu}_j(y_0))^{n^2}$. Since y_0 is an arbitrary element of Y the conditions of Lemma 10 are satisfied. Hence $\mu_j \in \Gamma(X)$.

Consider now the case where $qX = \{0\}$, where q is a prime number. Then also $qY = \{0\}$. Since the sets $\{\alpha_j\}$ and $\{\beta_j\}$ are admissible neither α_j nor β_j is divisible by q. Replacing the distribution μ_j by $\mu_j * \bar{\mu}_j$, we may assume that $\hat{\mu}_j(y) > 0$ for all $y \in Y$, $1 \leq j \leq s$. Consider now the contraction of Eq. (2) onto some subgroup $G \subset Y$ generated by an arbitrary element $y_0 \in Y$, $y_0 \neq 0$. Evidently $G \simeq \mathbf{Z}(q)$. It follows from Lemma 8 that $E = \left\{y \in G: \prod_{j=1}^{s} \hat{\mu}_j(y) = 1\right\} \neq \{0\}$. Hence $E = G$, i.e., $\hat{\mu}(y) \equiv 1$ on G, $1 \leq j \leq s$. This implies that $\hat{\mu}_j(y) \equiv 1$ on Y, i.e., $\mu_j \in \mathscr{D}(X) \subset \Gamma(X)$, $1 \leq j \leq s$. The theorem is thus proved.

Remark 1. From a complete description of groups X for which the Cramèr theorem on the decomposition of a Gaussian distribution is valid presented in paper [8] [the theorem states that if the sum $\xi_1 + \xi_2$ of independent random variables possesses a Gaussian distribution so do the components ξ_1 and ξ_2 (cf. Lemma 9)] it follows that this class of groups is substantially larger than the class for which the Skitovich–Darmois theorem is valid (cf. Theorem 1). The situation does not change if we require additionally that the characteristic functions $\hat{\mu}_j(y)$ of the distributions under consideration also satisfy the condition $\prod_{j=1}^{s} \hat{\mu}_j(y) \neq 0$ for all $y \in Y$ (cf. Theorem 3).

As in the case where $X = \mathbf{R}$ the Cramèr theorem on arbitrary groups follows from the following particular case of the Skitovich–Darmois theorem.

THEOREM B. Let $\xi_1, \xi_2, \xi_3, \xi_4$ be independent random variables with values in the group X and distributions $\mu_1, \mu_2, \mu_3, \mu_4$ and, moreover, let 1) $\mu_1 = \mu_3$, $\mu_2 = \mu_4$; 2) $\hat{\mu}_1(y) \times \hat{\mu}_2(y) \neq 0$ for any $y \in Y$. Then if the linear forms $L_1 = \xi_1 + \xi_2 + \xi_3 + \xi_4$ and $L_2 = \xi_1 + \xi_2 - \xi_3 - \xi_4$ are independent, $\mu_1, \mu_2 \in \Gamma(X)$.

Proof. Let ξ_1, ξ_2 be independent random variables with distributions μ_1, μ_2 and let the sum $\xi_1 + \xi_2$ possess the Gaussian distribution, i.e., $\mu_1 * \mu_2 \in \Gamma(X)$. Selecting ξ_3 and ξ_4 to have the same distribution as ξ_1 and ξ_2, respectively (the random variables obtained $\xi_1, \xi_2, \xi_3, \xi_4$ are independent) we observe that the linear forms $L_1 = (\xi_1 + \xi_2) + (\xi_3 + \xi_4)$ and $L_2 = (\xi_1 + \xi_2) - (\xi_3 + \xi_4)$ are independent. By Theorem B $\mu_1, \mu_2 \in \Gamma(X)$.

In turn, Theorem B follows from the Cramèr theorem. To prove this, we note that the independence of the linear forms L_1 and L_2 is equivalent to the fact that the characteristice functions $\hat{\mu}_1(y)$ and $\hat{\mu}_2(y)$ satisfy Eq. (7).

Set $\gamma = \mu_1 * \mu_2 * \bar{\mu}_1 * \bar{\mu}_2$. Then the characteristic function $\hat{\gamma}(y)$, as it follows from Eq. (7) satisfies the equation

$$\hat{\gamma}(u+v)\hat{\gamma}(u-v) = \hat{\gamma}^2(u)\hat{\gamma}^2(v), \quad u, v \in Y.$$

Condition 2 implies that $\hat{\gamma}(y) > 0$ for all $y \in Y$. Setting $\varphi(y) = \ln \hat{\gamma}(y)$, we observe (cf. definition 1) for $\gamma \in \Gamma(X)$. Then by Cramèr's theorem $\mu_1, \mu_2 \in \Gamma(X)$.

Thus the classes of groups on which the Cramèr theorem and Theorem B is valid are the same. If, however, in Theorem 2 we relinquish condition 2; then, as Lemma 3 indicates, an analogous assertion is not valid.

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DOMAINS OF VALUES OF SYSTEMS OF COEFFICIENTS OF BOUNDED TYPICALLY REAL FUNCTIONS IN THE DISK

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Let T_R denote the class of typically real functions in the disk $E = \{z: |z| < 1\}$, i.e., of the functions

$$F(z)=z+b_2z^2+\ldots+b_nz^n+\ldots \tag{1}$$

that are regular in E and satisfy in E the conditions $\operatorname{Im}F(z) = 0$ for $\operatorname{Im}z = 0$, $\operatorname{Im}F(z) \times \operatorname{Im}z > 0$ for $\operatorname{Im}z \neq 0$. Let $T_R(M)$ denote the class of the functions

$$f(z)=z+c_2z^2+\ldots+c_nz^n+\ldots\in T_R \tag{2}$$

such that $|f(z)| < M$ for $z \in E$, where M is a fixed number such that $1 < M < \infty$.

In [1] sharp estimates of the coefficients c_2, c_3, c_4 have been obtained with the help of integral representation in the class $T_R(M)$.

In the present article we find the domain of values of the system $\{c_2, \ldots, c_n\}$ in the class $T_R(M)$; the method to be used is based on the extremal properties of functions of the class T_R.

The following integral representation for the class of typically real functions is known [2; 3; 4, p. 516 and 517]: a function F(z) belongs to T_R if and only if it admits in the disk E the representation

$$F(z)=\int_{-1}^{1}\frac{z}{1-2tz+z^2}\,d\alpha(t), \tag{3}$$

where $\alpha(t)$ belongs to the class $A[-1, 1]$ of nondecreasing functions on $[-1, 1]$, such that $\alpha(1) - \alpha(-1) = 1$. Using the known (see [5, p. 161; 6, p. 172]) expansion (for $-1 \le t \le 1$)

$$\frac{z}{1-2tz+z^2}=\sum_{n=1}^{\infty}U_{n-1}(t)\,z^n,\quad z\in E, \tag{4}$$

from (4), (3), and (1) we get the following integral representation of coefficients for each function $F(z) \in T_R$:

$$b_n=\int_{-1}^{1}U_{n-1}(t)\,d\alpha(t),\quad n=1,2,\ldots. \tag{5}$$

In Eq. (4) and Eq. (5) the symbols $U_n(t)$, $t \in [-1, 1]$ denote Chebyshev polynomials of second order such that [5, p. 152; 6, p. 178]

$$U_0(t)=1,\ U_1(t)=2t,\ U_{n+1}(t)=2tU_n(t)-U_{n-1}(t),\ n=1,2,\ldots. \tag{6}$$

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