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# Nonlinear oscillations of topological structures in the sine-Gordon systems

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The nonlinear effect of the energy localization on topological inhomogeneities is investigated in the sine-Gordon systems. The regimes of nonlinear oscillations of nonequilibrium configurations of domain walls in the quasi-one-dimensional ferromagnet are described in terms of kink and breather solutions of the sine-Gordon equation. The conditions of the energy localization, i.e., the formation of breather excitations on these topological inhomogeneities, are found for the initial configurations of the dilated double kink structures. The results are obtained in the framework of the Schrödinger-type equation of the direct scattering problem associated with the sine-Gordon equation. It is shown that the final state of the evolution of the nonequilibrium topological spin structure represents the multi-frequency precessing domain wall in the ferromagnet, which radiates the continuous spectrum waves.

Keywords: nonlinear dynamics, sine-Gordon equation, ferromagnet, precessing domain wall, wobbling kink, radiation.

## Introduction

Topological structures in condensed matter physics are well known and presented by domain walls in magnets, dislocations in crystals, and fluxons in long Josephson junctions. They are described theoretically as the kink solutions in the framework of the corresponding nonlinear equations. In order to obtain explicit analytical results, the equations are reduced to the integrable one, usually one-dimensional sine-Gordon (SG) equation [1]. The latter possesses simple kink and antikink solutions, and their oscillating bound state, the breather, and, moreover, the exact multisoliton formula, which describes explicitly the stationary dynamics and interactions of all types of nonlinear and linear excitations of the sine-Gordon model.

In spite of integrability of the sine-Gordon equation, the analytical description of the non-stationary dynamics of kinks is not a simple task [2]. The analysis of a nonlinear stage of the evolution of nonequilibrium kink profiles can be performed after the solution of the direct scattering problem associated with the SG equation [2, 3]. It turned out [2] that nonequilibrium kinks can oscillate during a long time and weakly radiate linear waves, and generate the localized wave packets, and give birth to breathers.

Small stationary localized oscillations of the SG kink are absent, as well as corresponding frequencies of such internal modes in its linear excitation spectrum. However, nonlinear oscillations of the kink are possible and they correspond to the wobbling kinks or the wobbles [4, 5], which represents an oscillating topological structure formed by a kink and a breather settled on the top of the kink. The general pattern of the nonequilibrium kink evolution includes the process of the breather emergence and, hence, the corresponding energy localization on the kink, and the process of radiation of continuous spectrum waves, which carry away all the exceed energy [2].

The present study of the nonstationary kink dynamics in the sine-Gordon systems is aimed to show that the wobbling kink can be interpreted as the precessing domain wall in the one-dimensional anisotropic ferromagnet and to point out the conditions and methods of generation of such a nonlinearly oscillating topological structure. In this regard, it is interesting to notice that the existence and properties of the precessing domain wall in the one-dimensional anisotropic antiferromagnet have been already discussed in detail [6–8].

The paper is organized as follows. In the first section, we consider the model of the one-dimensional anisotropic ferromagnet and reduce its Hamiltonian to one of the discrete Takeno–Homma model. Then we obtain, traditionally in the long-wavelength limit, the double sine-Gordon and the sine-Gordon equation for the azimuthal angle variable of the chain spin rotation. Using the exact wobbler and kink solutions of these equations, we formulate the problem of the nonstationary evolution of the nonequilibrium spin topological structure to the 360° domain wall, which is stable in the magnetic field. In the next section, we find the conditions of the emergence of breathers by the nonequilibrium initial topological structure and determine the dynamical parameters and the energy of the precessing domain wall in a ferromagnet. Then we investigate the peculiarities of the process of radiation of the continuous spectrum waves and finally calculate the energy of the radiation that appears during the evolution of the double kink spin structure.

## Domain walls in anisotropic ferromagnetic chain in magnetic field

The quasi-one-dimensional ferromagnets, such as $CsNiF_3$ and TMNC [9, 10] can be considered above the 3D magnetic ordering temperature as a system of independent ferromagnetic chains with the Hamiltonian

$$\mathcal{H} = -J\sum_n \mathbf{S}_n\mathbf{S}_{n+1} + \frac{1}{2}\sum_n\left[D(S_n^z)^2 - A(S_n^x)^2\right] - g\mu_B H\sum_n S_x. \quad (1)$$

Here $\mathbf{S}_n = S_0(\sin\theta_n\cos\varphi_n, \sin\theta_n\sin\varphi_n, \cos\theta_n)$ is the classical spin with the value $S_0$ at the $n$th site, $\varphi_n$ and $\theta_n$ are the azimuthal and polar angles of the spin vector, respectively, $J$ is the exchange interaction constant, $A$ and $D$ are the easy-axis and easy-plane anisotropy constants, respectively, $H$ is the constant magnetic field, $g$ is the gyromagnetic ratio, and $\mu_B$ is the Bohr magneton.

In the case of the strong easy-plane anisotropy $D >> A$, when only a weak deviation of the spin vector from the easy plane is allowed

$$\theta_n = \frac{\pi}{2} - \vartheta_n, \quad \vartheta_n = \frac{g\mu_0 H}{DS_0}\frac{d\varphi_n}{d\tau} << 1, \quad (2)$$

we, following the scheme outlined in [11–13], approximately reduce the Hamiltonian (1) to the Hamiltonian of the Takeno–Homma model [14]:

$$\mathcal{H} = \frac{\hbar^2}{2DS_0^2}\sum_{n=1}^N \dot{\varphi}_n^2 - J\sum_{n=1}^N \cos(\varphi_n - \varphi_{n-1}) - \frac{1}{2}A\sum_{n=1}^N \cos^2\varphi_n - \frac{g\mu_0 H}{S_0}\sum_{n=1}^N \cos\varphi_n . \quad (3)$$

As seen from Eq. (3), such a model is described by only one scalar variable $\varphi_n$, and the point in the expression (4) means the differentiation with respect to time $\tau$. In the long-wavelength limit, when $J >> A$, from the Hamiltonian (3) we obtain the following dimensionless double sine-Gordon equation:

$$\varphi_{TT} - \varphi_{XX} + \sin\varphi\cos\varphi + h\sin\varphi = 0. \quad (4)$$

Here $T = \tau/\tau_0$ and $X = l_0 n$, where the unit of time is $\tau_0 = \hbar/(S_0\sqrt{DA})$ and $l_0 = a\sqrt{J/A}$ is the magnetic length, and the parameter of magnetic field is $h = g\mu_0 H / AS_0$.

In the case of the absence of magnetic field, $h = 0$, Eq. (4) is reduced to the usual integrable SG equation for the variable $u = 2\varphi$:

$$u_{TT} - u_{XX} + \sin u = 0. \quad (5)$$

Therefore, in terms of the azimuthal angle $\varphi$ there are the $\pi$-kink and $\pi$-antikink solutions of the SG equation, which correspond to domain walls of opposite signs:

$$\varphi_\pm(X) = 2\arctan\exp(\pm X). \quad (6)$$

In the nonzero magnetic field $h$, two identical domain walls form the static bound state, the wobbler [15, 16], which is exact solution of Eq. (4):

$$\varphi_W(X) = 2\arctan\exp(\kappa_W X - R_W) + 2\arctan\exp(\kappa_W X + R_W). \quad (7)$$

The parameters of the wobbler are the following functions of the magnetic field:

$$\kappa_W(h) = \sqrt{1+h}, \quad R_W(h) = \ln\left(\frac{\sqrt{1+h}-1}{\sqrt{h}}\right), \quad (8)$$

where $\kappa_W$ is the reverse effective length of the domain wall and $R_W$ is a half of the wobbler width. The wobbler configuration (7) means nothing but the 360° domain wall, the dimension of which is determined by the balance of the force of mutual repulsion of the identical 180° domain walls and the bringing-together action of the magnetic field.

When the field is small, $h << 1$, then $\kappa_W \approx 1$ and $R_W \approx \ln(\sqrt{h}/2)$. For example, for $h = 0.01$ the parameter $R_W$ is equal 3 and $\kappa_W$ is the unity with the accuracy 0.1 %, and the 180° domain walls are well separated and the distance between them is large enough in comparison with their effective lengths. Thus, in the small field, the 360° domain wall configuration has the form

$$\varphi_0(X) = 2\arctan\exp(X - R) + 2\arctan\exp(X + R). \quad (9)$$

In high fields, $h >> 1$, Eq. (4) is reduced to the SG equation

$$\varphi_{TT} - \varphi_{XX} + h\sin\varphi = 0. \quad (10)$$

This equation was first derived by Mikeska [17] to describe $2\pi$-solitons, i.e. the 360° domain walls, as nonlinear excitations in the quasi-one-dimensional easy-plane ferromagnet $CsNiF_3$. In the high-field limit in formulas (8) the parameter $\kappa_W \approx \sqrt{h}$ is large, and the parameter $R_W \approx 1/\sqrt{h}$ is small. Then the wobbler solution (7) is transformed in the following configuration:

$$\varphi_h(X) = 4\arctan\exp(\kappa_W X) - \frac{1}{h}\frac{\sinh(\kappa_W X)}{\cosh^2(\kappa_W X)}, \quad (11)$$

and in the main approximation the solution is reduced to the 2π-kink. After introducing the new time and coordinate variables $t = \sqrt{h}T$ and $x = \sqrt{h}X$, respectively, Eq. (10) takes the standard form of the SG equation

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0, \quad (12)$$

in which the well-known 2π-kink solution corresponds to the static 360°-domain wall

$$\varphi_{2\pi}(x) = 4\arctan\exp(x). \quad (13)$$

Note that the energy of the system in the dimensionless variables can be written as follows:

$$E = \int_{-\infty}^{\infty}\left(\frac{1}{2}\left(\varphi_t^2 + \varphi_x^2\right) + 1 - \cos\varphi\right)dx. \quad (14)$$

Now we are able to formulate the following nonstationary kink evolution problem: if we start from the wobbler spin configuration (7), prepared in the weak magnetic field $h_0$, and apply instantly the strong magnetic field $h$ to the ferromagnet, then how will this initial profile evolve and relax? The corresponding initial conditions for the problem in the framework of the SG equation (12) with the time and space variables, $t$ and $x$, looks like

$$\varphi(x,0) = 2\arctan\exp(\kappa x - R) + 2\arctan\exp(\kappa x + R), \quad \varphi_t(x,0) = 0, \quad (15)$$

where $\kappa = \sqrt{(1+h_0)/h}$ and $R = R_W(h_0)$ as follows from Eq. (8). Both parameters κ and $R$ can be considered as independent, because they are defined by two, initial and final, values of the magnetic field.

In theory, the homogeneous field applied to a whole ferromagnet has to form the identical wobblers in its chains. There is another possibility to prepare the initial structure (15) for the chosen 360° domain wall. It is enough to apply the local magnetic field, which is opposite to the homogeneous field $h$, to the spins arranged in and outside the central region the 2π-kink (13), in order to vary in such a manner the distance between two π-kinks in the initial configuration.

In the final stage of the evolution, the spin configuration (15) will relax to the 2π-kink (13), but there is the question of how the difference between energies of two topological structures (13) and (15) will be distributed. This problem is solved in the next section.

## Oscillating domain walls and breather birth in the sine-Gordon systems

The evolution problem of the initially static wobbler-like structure (15) in the framework of the SG Eq. (12) can be considered following the scheme presented in Ref. 2. We study this question by solving the direct scattering problem associated with the SG equation (12), which can be formulated as the following matrix eigenvalue problem [2, 3]:

$$i\mathbf{J}_x = \begin{bmatrix} S_-\cos\left(\frac{\varphi}{2}\right) & -S_+\sin\left(\frac{\varphi}{2}\right) \\ -S_+\sin\left(\frac{\varphi}{2}\right) & -S_-\cos\left(\frac{\varphi}{2}\right) \end{bmatrix}\mathbf{J}. \quad (16)$$

Here $\mathbf{J} = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$ is the Jost functions, and the matrix contains initial spin configuration (15), $z = 2\lambda$ is the doubled spectral parameter, which can be represented for breathers as $z = \exp(i\chi)$, and the parameters $S_\pm$ are expressed as follows:

$$S_+ = \frac{1}{2}\cos\chi, \quad S_- = \frac{i}{2}\sin\chi. \quad (17)$$

The frequency of the breather is equal to $\omega = |\cos\chi|$ and the parameter ε, which is given as $\varepsilon = |\sin\chi|$, defines the amplitude and the reverse width of the breather and, eventually, its energy $E_{\text{br}} = 16\varepsilon$ [1]. In Ref. 2, it is shown that the spectral problem (16) can be reduced to the eigenvalue problem of one-dimensional Schrödinger operator

$$\left(-\frac{d^2}{dx^2} + W(x,\varepsilon)\right)f = 0, \quad (18)$$

where the eigenfunction $f$ is connected with $\psi_1$ as follows:

$$f = \mathrm{Re}(\psi_1)\Big/\sqrt{S_+\sin\frac{\varphi}{2}}, \quad (19)$$

and the potential well is expressed explicitly through φ and depends on the parameter ε:

$$W(x,\varepsilon) = \frac{1}{4}\left(\left(\frac{d\ln\sin\frac{\varphi}{2}}{dx}\right)^2 - 2\frac{d^2\ln\sin\frac{\varphi}{2}}{dx^2} - \sin^2\frac{\varphi}{2} + 2\varepsilon\cos\frac{\varphi}{2}\frac{d\ln\tan\frac{\varphi}{2}}{dx} + \varepsilon^2\right). \quad (20)$$

Returning to the physical problem on the evolution and relaxation of the nonequilibrium spin structure (15), at first we consider the case of strong magnetic fields. If the initial field is strong enough, $h_{\text{in}} >> 1$, then we can retain only the main part of the solution (11) and take it in the form

$$\varphi = 4\arctan\exp(\kappa x),$$
$$\sin\frac{\varphi}{2} = \frac{1}{\cosh\kappa x}, \quad \cos\frac{\varphi}{2} = -\tanh\kappa x. \tag{21}$$

The parameter $\kappa$ is defined by the ratio of the initial field to the present one as $\kappa = \sqrt{h_{\rm in}/h}$. The corresponding eigenvalue problem was solved in Ref. 2. The Schrödinger-like equation takes the following form:

$$\left\{-\frac{d^2}{dx^2} - \frac{1}{4}\left(1-\kappa^2\right)\frac{1}{\cosh^2\kappa x}\right\} f = -\frac{1}{4}(\kappa+\varepsilon)^2 f. \tag{22}$$

If we start from the field $h_{\rm in} < h$, then the parameter $\kappa$ becomes less than the unity, and in the case $\kappa \le 1/2$ the discrete spectrum of eigenvalues arises [18], and hence, the breather birth occurs. The breather parameter $\varepsilon_n$ is given by the formula

$$\varepsilon_n = 1 - 2\kappa n, \tag{23}$$

where integer $n$ numbers breathers. As seen from Eq. (23), the first breather appears when $\kappa = 1/2$, i.e., $h = 4h_{\rm in}$. In Ref. 2 it is shown that when $\kappa = 1/3$, and hence, $h = 9h_{\rm in}$, then the energy of the born breather reaches maximum and the radiation of continuous spectrum waves disappears. The corresponding initial profile is transformed in the wobbling kink, which oscillates with the frequency of breather $\omega = 2\sqrt{2}/3$ and describes the precessing domain wall in the ferromagnet. The energy of the radiation, represented in [2], looks like the weakly decaying spike-like structure with strictly periodic positions of maxima at the breather birth points and minima corresponding to the radiation energy zeroes.

In the present study, we concentrate on the evolution of the topological structure which has the form (15) supposing that $\kappa = 1$ and $R$ is a free parameter. As pointed out earlier, we can apply the local magnetic field to the central region of the domain structure to expand it keeping in mind that the effective width of $\pi$-kinks remains unchanging. Thus, we aim to distinguish the effect of the structure width on the process of the breather emergence.

At first notice that the expression (15) with $\kappa = 1$ can be rewritten as

$$\varphi(x,0) = \pi + 2\arctan\left(\frac{\sinh x}{\cosh R}\right). \tag{24}$$

After substituting this initial condition to Eq. (20), we obtain the explicit form of the potential for the Schrödinger operator (18)

$$W_R(x) = \frac{1}{4}\left(1 + \frac{2\sinh^2 R}{\cosh 2x + \cosh 2R} - \frac{3\sinh^2 2R}{(\cosh 2x + \cosh 2R)^2} + \frac{2\sqrt{2}\varepsilon\cosh x}{\sqrt{\cosh 2x + \cosh 2R}} + \varepsilon^2\right). \tag{25}$$

In order to reveal the values of the parameter $R$, when breathers appear, we notice that the parameter $\varepsilon = 0$ at these points. Then the potential (25) is simplified and takes the form

$$W_R^0(x) = \frac{1}{4}\left(1 + \frac{2\sinh^2 R}{\cosh 2x + \cosh 2R} - \frac{3\sinh^2 2R}{(\cosh 2x + \cosh 2R)^2}\right). \tag{26}$$

The corresponding Schrödinger equation with the potential well (26) is solved numerically, and we find the following sequence of values of the parameter $R$ when a new breather appears: $R_1 = 2.424$, $R_2 = 5.59$, $R_3 = 8.732$ and so on. Then we solve the corresponding Schrödinger Eq. (18) with the full potential (25), preliminarily transforming it into the form

$$\left(-\frac{d^2}{dx^2} + V(x,\varepsilon)\right) f = -\frac{1}{4}\varepsilon(1+\varepsilon) f, \tag{27}$$

where the potential $V(x,\varepsilon)$ tends to the constant value $V_0 = 1/4$ at infinity for all $\varepsilon$:

$$V(x,\varepsilon) = \frac{1}{4}\left(1 + \frac{2\sinh^2 R}{\cosh 2x + \cosh 2R} - \frac{3\sinh^2 2R}{(\cosh 2x + \cosh 2R)^2} - 2\sqrt{2}\varepsilon\left(1 - \frac{\cosh x}{\sqrt{\cosh 2x + \cosh 2R}}\right)\right). \tag{28}$$

For this rewritten form of the potential well, all new levels are detached below from zero. In Fig. 1, we show the potential well with discrete levels for two values of the parameter $R$. As seen from Fig. 1(a), there are two levels and two breathers respectively in the case $R = 5.75$ and three levels and three breathers in the case $R = 8.75$, at that the third level and, hence, the third breather, as pointed out above, just appeared at $R = R_3$.

In the issue we have calculated all the discrete eigenvalues $\varepsilon_n$ of the Schrödinger equation (27) as functions of $R$ in the wide range of the parameter. We present these results in Fig. 2 as dependencies of breather energies $E_{\rm br}(R) = 16\varepsilon_n(R)$.

Further, we calculate the energy (14) of the initial topological structure (24) and find the expression:

$$E_{\rm in}(R) = 2 + 2R\left(\frac{3}{\tanh R} - \tanh R\right). \tag{29}$$

When $R = 0$, the structure (24) is transformed in the $2\pi$-kink, and then its energy is equal to $E_0 = 8$. In the case of large $R$, the energy behaves as the linear function of this parameter.

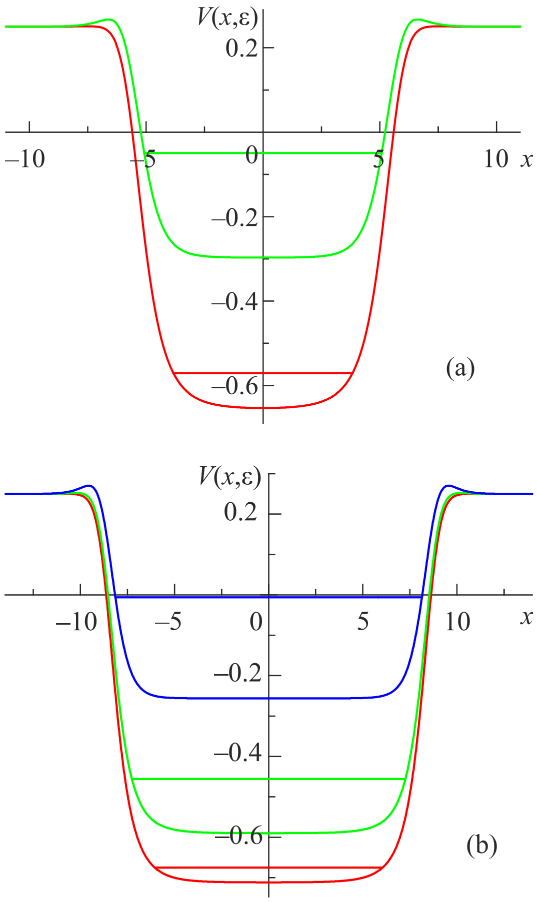


*Fig. 1.* Potential wells with discrete levels for the Schrödinger equation (27): there exist two levels in the case $R = 5.75$ (a) and three levels in the case $R = 8.75$ (b).

Choosing the fixed value $R_*$ in Fig. 2 and summing all $E_{\text{br}}(R_*)$ from different branches, we find the total energy of breathers $E_{\text{tot}}(R_*)$. The difference between $E_{\text{in}}(R_*)$ and the sum $8 + E_{\text{tot}}(R_*)$ of energies of nonlinear excitations, $2\pi$-kink and breathers, gives $E_r(R_*)$, the certain value of the radiation energy of the continuous spectrum waves. As a result, we calculate the full radiation energy dependence on the parameter $R$ and present it in Fig. 3.

Now we are able to imagine the evolution process of the nonequilibrium 360° domain wall structure. Being initially static, the spin structure begins to precess in the central part of the domain and to generate the continuous spectrum waves. At the same time, if the condition of breather birth fulfills, then the precession in the domain does not decay, and a part of the stored energy is localized on the domain. As seen from Fig. 3, the energy, carried out by the continuous spectrum waves, has maxima at points where a new breather is emerged by oscillating domain wall structure. On the other hand, there are values of the parameter $R$ when the radiation is minimal, and the energy is mainly confined by the domain structure. As it has found by authors in Ref. 2 and pointed out above, the spike-like form of the radiation energy dependence takes place for the initial spin structure (21). We note now that such behavior is the common feature of the evolution of the nonequilibrium domain structure and does not reduce only to some kind of the size effect. The birth of breathers plays here the crucial role.

Thus, the dilation of the domain structure creates the nonequilibrium configuration that can evolve with emerging breathers. On the contrary, the compressed structure expands, dumping the exceed energy through radiation of the continuous spectrum waves. The simpler example of the evolution of the nonequilibrium structure (21) confirms this conclusion because for values of the parameter $\kappa > 1/2$ no breather can appear from this initial configuration.

In Ref. 2, the double kink ansatz was discussed, which described the compressed structure of a different type than the solution (21). It has the following form:

$$\varphi(x) = 2\left(\arctan\exp\left(x + i\delta\right) + \arctan\exp\left(x - i\delta\right)\right) \quad (30)$$

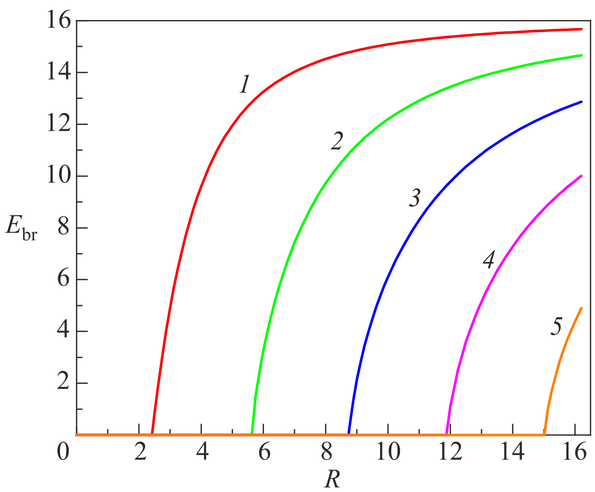


*Fig. 2.* The energies of breathers as functions of the parameter $R$ of the initial spin structure.

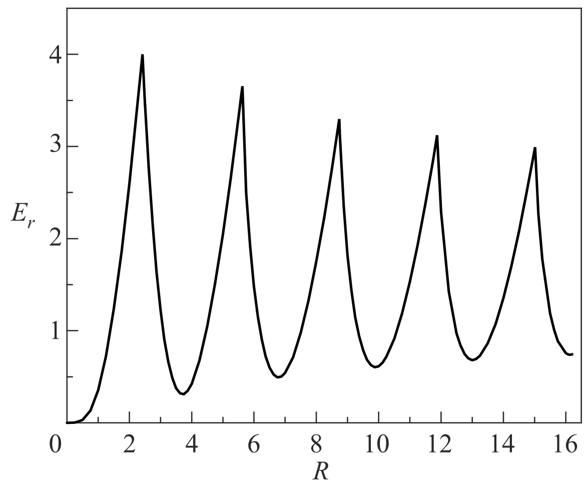


*Fig. 3.* The radiation energy dependence on the parameter $R$.

and, being the real function, it can be written as

$$\varphi(x) = \pi + 2\arctan\left(\frac{\sinh x}{\cos\delta}\right). \qquad (31)$$

We used this function as the initial condition in the SG equation and checked the eigenvalue problem (18) with the corresponding potential well

$$W_\delta(x) = \frac{1}{4}\left(1 - \frac{2\sin^2\delta}{\cosh 2x + \cos 2\delta} + \right.$$
$$\left. + \frac{3\sin^2 2\delta}{(\cosh 2x + \cosh 2\delta)^2} + \frac{2\sqrt{2}\varepsilon\cosh x}{\sqrt{\cosh 2x + \cos 2\delta}} + \varepsilon^2\right), \qquad (32)$$

and no breather solutions and appropriate solutions of the spectral problem were revealed.

Ending this section, we conclude that, extended domain wall structure, preliminarily enlarged by the homogeneous or local auxiliary field, can evolve and relax with emerging the breathers and arising the multi-breather wobbling kink, or, in terms of magnetism, with the formation of the multi-frequency precessing domain wall in a quasi-one-dimensional ferromagnet. In the next section, we present numerical results of simulation of nonlinear dynamics of the sine-Gordon equation, which confirm analytical findings set out above.

## Numerical simulation of the evolution of extended domain wall structures

The solving of the direct scattering problem associated with the SG equation predicts the results of the numerical simulation of nonlinear dynamics of this system. We use the typical difference scheme of the integration of the SG equation, starting from the initial conditions formulated in the previous sections. The obtained results are presented in the following figures.

At first, we simulate the evolution of the initial domain structure with the parameter $R = 1.75$, which is less than the critical value $R_1$, when the first breather is born. The results for a half of the period of oscillations of the evolving structure, beginning from time $t = 240$, are shown in Fig. 4. The small radiation of almost linear waves by the oscillating domain wall is observed. These results are in full agreement with those predicted theoretically in Ref. 2.

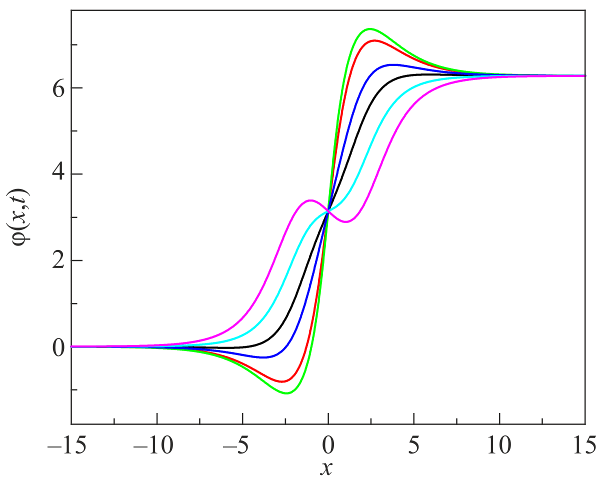


*Fig. 5.* The one-half period sequence of profiles of the oscillating 360° domain wall evolved from the initial structure with the parameter $R = 4$.

As seen in Fig. 2, the amplitude and the energy of the first breather rise sharply with the increase of the parameter $R$ after the appearance at the point $R_1$. In Fig. 5, the oscillating 360° domain wall is shown, which has evolved from the initial structure with the parameter $R = 4$. It is remarkable that in this case the pure wobbling kink is formed, and any notable radiation is not observed near the kink. This is the result of the previous emergence of the perfectly localized wave packets by the oscillating kink. The large wave packet and running ahead small one are shown in Fig. 6.

We have studied nonlinear oscillations of the obtained precessing domain wall by spectral analysis of the time

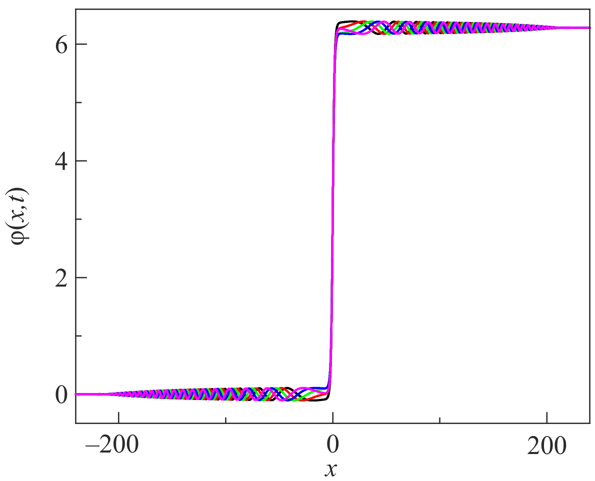


*Fig. 4.* The one-half period sequence of the 360° domain wall profiles with small radiated linear waves for the case of the initial structure with the parameter $R = 1.75$.

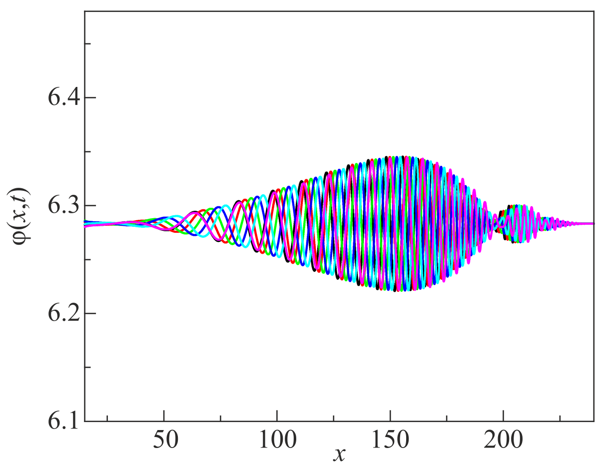


*Fig. 6.* The one-half period sequence of profiles of the localized wave packets radiated by the oscillating domain structure in the case of the parameter $R = 4$.

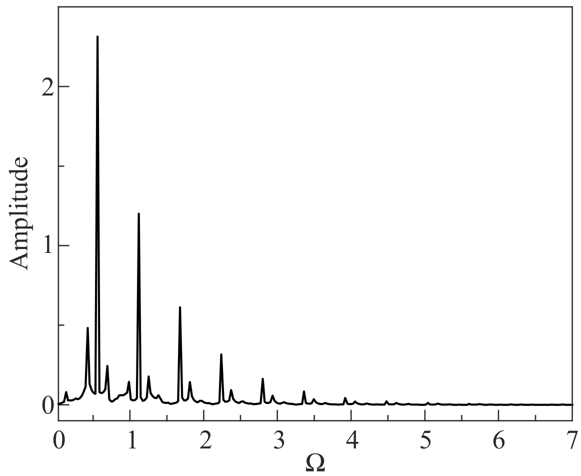


*Fig. 7.* The frequency spectrum of nonlinear oscillations of the domain wall in the case of the parameter $R = 4$.

series of the kink slope motion. The frequency spectrum is shown in Fig. 7. The sequence of the larger peaks corresponds to the main and higher multiple harmonics of the breather, and the largest one coincides perfectly with the frequency obtained from the discrete spectrum of the Schrödinger operator (18) in the case of the parameter $R = 4$.

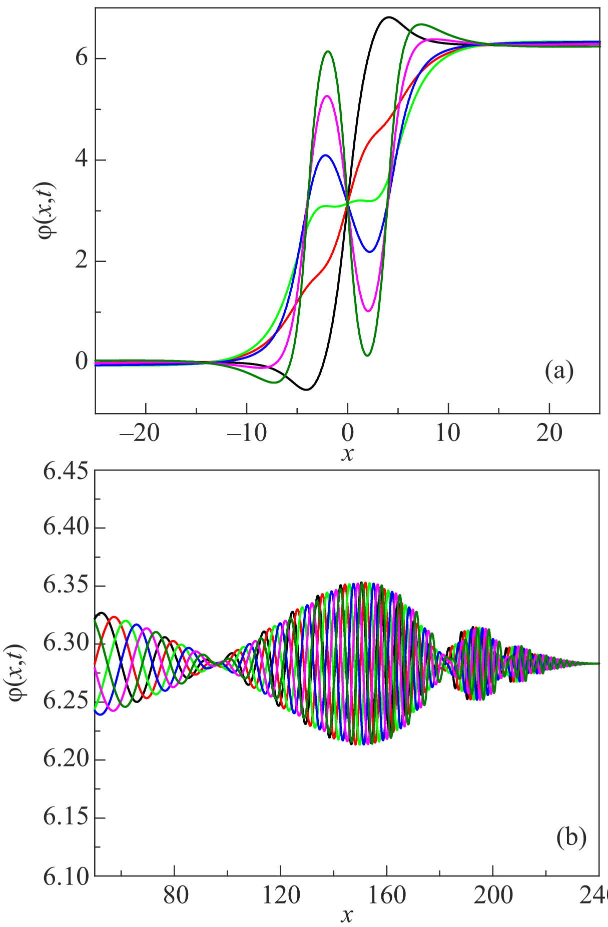


*Fig. 8.* The one-half period sequence of profiles of the oscillating 360° domain wall evolved from the initial structure with the parameter $R = 6.625$ (a). The one-half period sequence of profiles of the wave packets radiated by the domain wall (b).

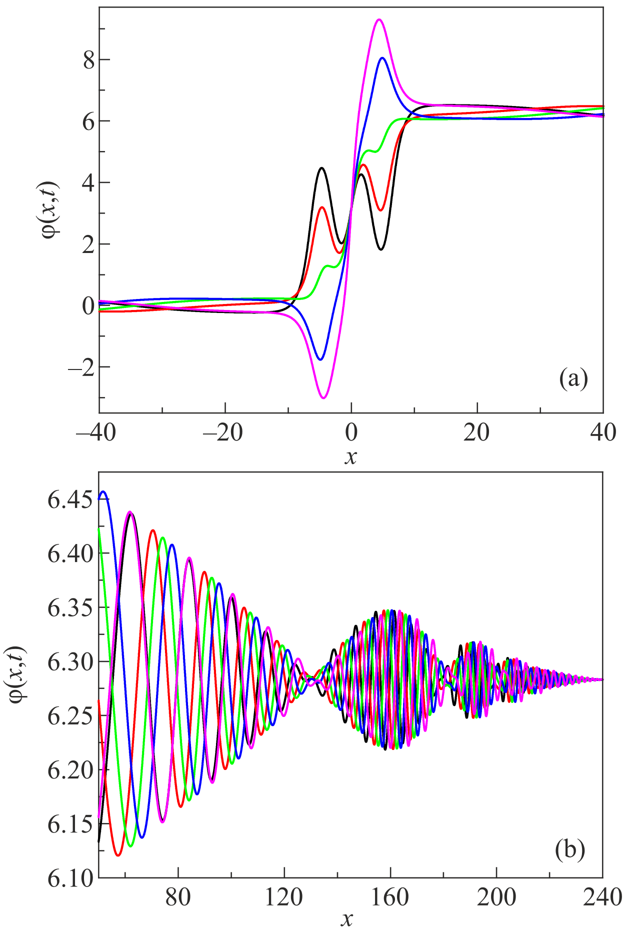


*Fig. 9.* The one-half period sequence of profiles of the oscillating 360° domain wall evolved from the initial structure with the parameter $R = 8.75$ (a). The one-half period sequence of profiles of the wave packets radiated by the domain wall (b).

The multi-frequency wobbling kinks for large values of the parameter $R$ are presented in Figs. 8(a) and 9(a). They are well-separated from the large wave packets, radiated at an earlier stage of the evolution of the initial domain structure. These wave packets are shown in Figs. 8(b) and 9(b). Note the tendency of structuring the radiation with the formation of the sequence of the fast-traveling well-localized smaller wave packets from the largest one.

Thus, all obtained numerical results are in full agreement with the predictions of the above proposed theory.

## Conclusion

Nonstationary dynamics of nonequilibrium domain wall structures in the quasi-one-dimensional ferromagnet was investigated in the framework of the sine-Gordon equation. Oscillations of the topological spin structures were described in terms of nonlinear excitations of the sine-Gordon equation, kinks and breathers. Theoretical analysis was performed by using the authors' reformulation of the direct scattering method approach as the spectral problem of the one-dimensional Schrödinger operator with the potential well defined the initial spin configuration. By exact solving the eigenvalue problem for the operator, the conditions of the emergence of breathers by the nonequilibrium topological spin structure were found, and dynamical parameters and the energy of the precessing domain wall in the ferromagnet were determined. For initial configurations of the dilated double kink structures, corresponding the stretched 360° domain wall, the energies of the born breathers were calculated, and the energy of radiation, which was generated during the evolution of the nonequilibrium structure, was found. The dependence of the radiation on the parameter of the width of the initial structure was obtained. It had the almost periodic form with spike-like peaks corresponding to the points of a new breather birth and minima for those values of the dimension of the initial structure when the stored energy mainly remained in the nonlinearly oscillating central domain. Numerical simulation of the nonlinear dynamics of the sine-Gordon equation with the corresponding initial conditions entire confirmed the theoretical predictions and additionally gave a possibility to investigate the peculiarities of well-localized wave packet formation in the process of the radiation of continuous spectrum waves.

At last, we note that the obtained results can be applied to other physical systems, which are described by the sine-Gordon model. First of all, it concerns the long Josephson junctions and their fluxon dynamics.

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1. S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons*: *The Inverse Scattering Method,* Springer US, New York (1984).
2. M. M. Bogdan and O. V. Charkina, *Fiz. Nizk. Temp.* **47**, 173 (2021) [*Low Temp. Phys*. **47**, 155 (2021)].
3. R. Buckingham and P. D. Miller, *Physica D* **237,** 2296 (2008).
4. H. Segur, *J. Math. Phys.* **24**, 1439 (1983).
5. G. Kalbermann, *J. Phys. A: Math. Gen.* **37**, 11603 (2004).
6. I. V. Baryakhtar and B. A. Ivanov, *Fiz. Nizk. Temp.* **5**, 759 (1979) [*Sov. J. Low Temp. Phys.* **5**, 361 (1979)].
7. B. A. Ivanov, A. K. Kolezhuk, *Fiz. Nizk. Temp.* **21**, 355 (1995) [*Low Temp. Phys*. **21**, 275 (1995)].
8. M. M. Bogdan and O. V. Charkina, *Fiz. Nizk. Temp.* **40**, 105 (2014) [*Low Temp. Phys*. **40**, 84 (2014)].
9. C. Dupas and J. P. Renard, *J. Phys. C* **10,** 5057 (1977).
10. H.-J. Mikeska and M. Steiner, *Adv. Phys.* **40**, 191 (1991).
11. M. V. Gvozdikova and A. S. Kovalev *Fiz. Nizk. Temp.* **25**, 1295 (1999) [*Low Temp. Phys.* **25**, 972 (1999)].
12. O. V. Charkina and M. M. Bogdan, *Fiz. Nizk. Temp.* **44**, 824 (2018) [*Low Temp. Phys.* **44**, 644 (2018)].
13. M. M. Bogdan, V. I. Belan, and O. V. Charkina. *Fiz. Nizk. Temp.* **44**, 1700 (2018) [*Low Temp. Phys.* **44**, 1331 (2018)].
14. S. Homma and S. Takeno, *Prog. Theor.Phys.* **72**, 679 (1984).
15. D. K. Campbell, M. Peyrard, and P. Sodano, *Physica D* **19**, 165 (1986).
16. M. M. Bogdan, A. M. Kosevich, and G. A. Maugin, *Wave Motion* **34**, 1 (2001).
17. H.-J. Mikeska, *J. Phys. C: Solid State Phys.* **11** L29 (1978).
18. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Pergamon, New York (1977).


## Нелінійні коливання топологічних структур в системах синус-Гордон

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Досліджено нелінійний ефект локалізації енергії на топологічних неоднорідностях в системах синус-Гордон. Режими нелінійних коливань нерівноважних конфігурацій доменних стінок у квазіодновимірному феромагнетику описуються в термінах кінкових та бризерних розв’язків рівняння синус-Гордона. Знайдено умови локалізації енергії, тобто формування бризерних збуджень на цих топологічних неоднорідностях, для початкових конфігурацій розтягнутих подвійних кінкових структур. Результати отримано за допомогою рівняння типу Шредінгера в рамках метода оберненої задачі розсіювання, пов’язаної з рівнянням синус-Гордона. Показано, що кінцевим станом еволюції нерівноважної топологічної спінової структури є багаточастотно осцилююча доменна стінка, що прецесує у феромагнетику, і випромінює хвилі безперервного спектра.

Ключові слова: нелінійна динаміка, рівняння синус-Гордона, феромагнетик, доменна стінка, що прецесує, воблінг кінк, випромінювання.