

# Soliton complex dynamics in strongly dispersive medium

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## Abstract

The concept of soliton complex in a nonlinear dispersive medium is proposed. It is shown that strongly interacting identical topological solitons in the medium can form bound soliton complexes which move without radiation. This phenomenon is considered to be universal and applicable to various physical systems. The soliton complex and its “excited” states are described analytically and numerically as solutions of nonlinear dispersive equations with the fourth and higher order spatial or mixed derivatives. The dispersive sine-Gordon (dSG), double and triple sine-Gordon, and piecewise linear models are studied in detail. Mechanisms and conditions of the formation of soliton complexes, and peculiarities of their stationary dynamics are investigated. A phenomenological approach to the description of the complexes and the classification of all the possible complex states are proposed. Some examples of physical systems, where the phenomenon can be experimentally observed, are briefly discussed. © 2001 Elsevier Science B.V. All rights reserved.

*Keywords:* Soliton complex; Dispersive medium; Soliton interaction; Radiationless motion

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## 1. Introduction

Properties of wave excitations in condensed matter are strongly influenced by spatial dispersion. In solids dispersion originates from the discreteness of real crystals. The nonlinear dynamics of lattice models exhibits, thus, many specific phenomena known as the discreteness effects [1–7]. In the long wave limit, when the systems are considered as continuous, these effects have to disappear. However, some of them leave a trace in the continuum limit, and hence can be picked as a class of universal dispersive effects.

Naturally the same situation is observed in macroscopic discrete systems, e.g. arrays of Josephson junctions and nonlinear transmission lines, systems with nonlocal interactions, and others [8–15].

The universal effects can also manifest themselves in strongly dispersive media, in which the dispersion is not a consequence of translational symmetry of underlying structures. Examples of such systems are plasmas, fluids, optical and dissipative–dispersive systems [16–23].

One of these effects, occurring in the dynamics of nonlinear excitations of a dispersive medium is the formation of bound states of solitons [9,24]. The role of dispersion as a factor influencing the interaction between well-separated solitons was discussed in a great number of works (see, e.g. [9,18,22–24]). As a result, two mechanisms of a formation of soliton bound states were found out. In the first case, interactions of oscillating soliton tails in dispersive media

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lead to the formation of bunches of solitons consisting of two or more well-distinguishable humps [18,22]. In the second case, solitons coexisting with resonant radiation can form bound states with purely solitonic asymptotics due to some kind of the radiation interference effect [25–27]. In these theories, the contribution of the dispersion to the soliton interaction has to be considered as a weak perturbation. As a result, the energy of the formed multisoliton structure differs slightly from the energy of a corresponding set of the noninteracting solitons.

In the case of topological solitons, this phenomenon of soliton bunching was observed through numerical simulations beginning with the work [8], where, in fact, the weakly discrete sine-Gordon model was studied. Then there were attempts [28,29] at explaining the effect based on the use of the soliton perturbation theory applied to the continuous analogue of the system. McLaughlin and Scott [28] were the first who pointed out that any dispersive terms should be added to the usual sine-Gordon model to obtain the multisoliton steady solutions. Then Peyrard and Kruskal observed by means of a numerical simulation the almost radiationless motion of the  $4\pi$ -soliton in the highly discrete sine-Gordon system [2]. This effect occurs not only in the discrete model [2,8,15] but also in the continuous dispersive sine-Gordon equation, and a corresponding analytical solution for the  $4\pi$ -soliton complex can be found exactly [30,31].

In general, topological multisolitons exist as discrete sets of solitonic configurations with internal structures. They are well-studied in systems with nonlocal interactions [11,12,14]. At last, bound solitons can also be realized in systems of anharmonically interacting particles. Such models are described by the Boussinesq-type equations with the high spatial derivatives [22,32,33], and properties of the bound solitons in these dispersive systems were studied in detail in both cases of topological and non-topological solitons [22,32–37].

In the present paper, we concentrate just on the case of strong interaction between topological solitons. This occurs when identical solitons are closely placed. As a result, the repulsive potential of the solitons has to grow rapidly with decreasing the distance between them. This causes the applicability of the perturbation theory, basing on the “one-soliton” approximation to become invalid. Then one has to take into account the dispersion, as an additional source of strong interaction, and this can change the character of the interaction between solitons in general.

The aim of this paper is to show that solitons in a strongly dispersive medium possess an internal structure and their interaction depends on intrinsic properties such as flexibility. Due to this dependence, the potential energy turns out to be a non-monotonic function of the distance between solitons. As a result, identical solitons can attract each other and form a bound soliton complex which can move without any radiation in strongly dispersive media [30,31]. Thus, we call the bound soliton states with zero and small distances between composite solitons, the soliton complex and its “excited” states, respectively.

We present a number of dispersive models bearing such topological soliton complexes. The models are described by nonlinear equations with fourth and higher spatial or mixed derivatives. Solutions of the relevant equations can be obtained numerically and analytically. We found exact analytical solutions for two variants of the dispersive sine-Gordon (dSG) and double sine-Gordon equations (dDSG). We also show that the complex, consisting of three solitons can be described explicitly in the dispersive equations with the additional sixth derivative in the case of the sine-Gordon and triple sine-Gordon models.

We propose the classification of the “excited” states of soliton complexes, constructing them explicitly in the framework of the double piecewise-linear dispersive model. The two-soliton ansatz approximation is used to establish analytically the existence condition for the soliton complex in the dSG and dDSG equations. We find numerical solutions of these equations for the two-soliton complex and its “excited” states, and the dependence of their energies and velocities on the model parameters. As a result, we formulate the concept of the soliton complex and classify it as a specific bound state of strongly interacting identical solitons in a dispersive medium.

## 2. Dispersive models with soliton complex

As a first example of a dispersive model, we mention the discrete sine-Gordon system which is described by the equation [2,4,38]:

$$\frac{\partial^2 u_n}{\partial \tau^2} + 2u_n - u_{n-1} - u_{n+1} + \frac{1}{d^2} \sin(u_n) = 0, \tag{1}$$

where  $u_n$  is, e.g. the displacement of atom  $n$  and  $d$  is the discreteness parameter. A stationary motion of a single  $2\pi$ -soliton is impossible in this dispersive system because of a strong radiation emitted by the moving  $2\pi$ -soliton [4]. At the same time, numerical simulations [2,8,15] showed almost radiationless motion of the  $4\pi$ -soliton and other  $2m\pi$ -soliton complexes. Peyrard and Kruskal [2] tried to explain the formation of the soliton complex of two identical  $2\pi$ -solitons by exploiting the fact of the presence of the Peierls potential in the lattice under consideration.

However, in works [30,31] it was found that the radiationless motion of the complex can be described explicitly in the framework of the dispersive sine-Gordon equation (1dSG) with a fourth spatial derivative:

$$u_{tt} - u_{xx} - \beta u_{xxxx} + \sin(u) = 0. \tag{2}$$

Eq. (2) is obtained as the long wave limit of Eq. (1) by substituting  $xd$  for  $n$ ,  $td$  for  $\tau$ , and the second difference by the series:

$$u_{n-1} + u_{n+1} - 2u_n \approx u_{xx} + \beta u_{xxxx} + \dots \tag{3}$$

The relation between the dispersive factor  $\beta$  and the discreteness parameter  $d$  is  $\beta \equiv 1/(12d^2)$ . The exact solution of Eq. (2), corresponding to the soliton complex, has the following form:

$$u_{4\pi} = 8 \arctan \left\{ \exp \left( \sqrt{\frac{2}{3}} \frac{(x - V_0 t)}{\sqrt{1 - V_0^2}} \right) \right\}, \tag{4}$$

$$V_0 = \pm \sqrt{1 - \sqrt{\frac{4}{3}} \beta}. \tag{5}$$

The velocity of the complex is not an arbitrary constant in the solution (Eq.(4)), but it is a function of the dispersive parameter  $\beta$ . As a function of the discreteness parameter, it equals  $V_0(d) = \pm \sqrt{1 - (1/3d)}$  and differs less than 5% from numerical result of Peyrard and Kruskal, as it follows from a comparison of Fig. 1 (solid line) and Fig. 12 of [2]. This fact provides the starting point of our investigation because it shows that taking into account a high

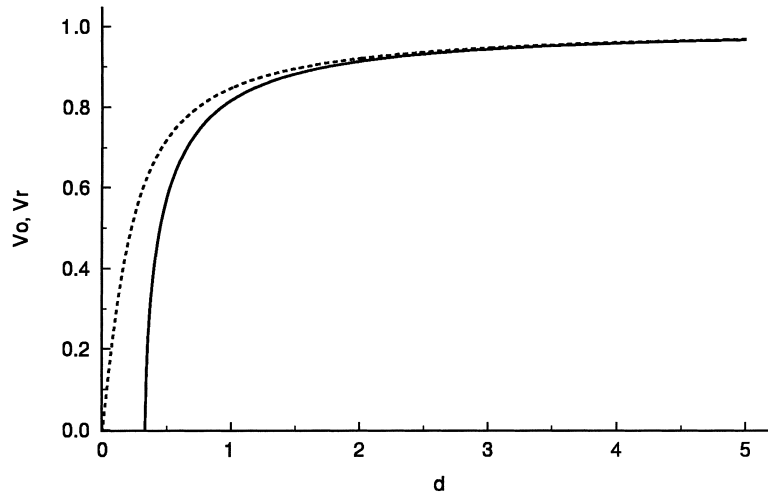


Fig. 1. Velocities of soliton complexes as functions of the discreteness parameter  $d$ . Solid (dash) line corresponds to the 1dSG (2dSG) equation.

order dispersion in the continuum model described by Eq. (2), leads to the same phenomenon as that in the discrete model of Eq. (1).

From the other hand, it is clear that the continuum model of Eq. (2) describes properly stationary moving nonlinear excitations in the original discrete model of Eq. (1) only in the limit  $\beta \ll 1$ . In this case using Eq. (4), one can write the discrete  $4\pi$ -soliton in the first approximation as

$$u_n \simeq 8 \arctan \left\{ \exp \left[ \sqrt{\frac{2}{d}} (n - V_0 \tau) \right] \right\}.$$

For small  $\beta$  (large  $d$ ), the solution is a smoothly-varying function of number  $n$ , and its effective width is proportional to  $\sqrt{d}$ . This provides a validity of the continuum approximation. It is remarkable and surprising that the analytical expression  $V_0(d)$  for a velocity holds good in the highly discrete system, i.e. far from the continuum limit. It would be also noted that the  $4\pi$ -soliton appears to be localized more strongly than the  $2\pi$ -kink of the usual sine-Gordon equation, since the kink width is proportional  $d$ .

It is evident [31] that solutions similar to Eq. (4) are available for the second or “regularized” dispersive sine-Gordon equation (2dSG) which has the fourth spatio-temporal mixed derivative instead the fourth spatial derivative of Eq. (2):

$$u_{tt} - u_{xx} - \beta u_{ttxx} + \sin(u) = 0. \quad (6)$$

The form of the soliton complex solution of the equation is the same as for the dSG equation, however, the velocity dependence on the parameter  $\beta$  differs from Eq. (5).

$$V_r(\beta) = \pm \left( \sqrt{1 + \frac{1}{3}\beta} - \sqrt{\frac{1}{3}\beta} \right). \quad (7)$$

As a function of the parameter  $d$ , the velocity is also shown in Fig. 1 (dash line).

The soliton complex with the internal structure was virtually identified by Peyrard et al. [39] in simulations of the continuous modified sine-Gordon model. In general, the taking into account the spatial dispersion or nonlocal interactions leads to a possibility of the complex formation [40]. In fact, the presence of freely moving soliton complexes and their “excited states” was examined numerically in the continuous nonlocal sine-Gordon models in works [11,12,14]. Such models are described by integro–differential equations which can be reduced to systems of two local equations in the case of exponentially-decaying kernel. As an example, we point out the equation [11]:

$$u_{tt} + \sin(u) = \frac{\partial}{\partial x} \int_{-\infty}^{\infty} dx' G(x - x') u_{x'}(x', t). \quad (8)$$

In the case of the kernel  $G(x, \lambda) = (1/2\lambda) \exp(-|x|/\lambda)$ , this equation is transformed into the set of equations

$$\lambda^2 w_{xx} - w = -u_x, \quad u_{tt} + \sin(u) = w_x,$$

which possess soliton complex solutions.

All the above facts collected together prompt that the formation of the soliton complexes is a universal property of the strongly dispersive media. Therefore, one can try to find solutions for the soliton complexes in more general systems than the dispersive sine-Gordon models.

The present paper is mainly devoted to studying the soliton complex in the dDSG equations because of two reasons. First, the usual DSG equations describe a large variety of physical systems: ferro- and antiferro-magnets, magneto–elastic systems, superfluid  $^3\text{He}$  and others [41–43]. Secondly, in the usual DSG equation,  $2\pi$ -kinks form the  $4\pi$ -kink, the wobbler, due to the action of an external field. In the dispersive DSG equations both factors, dispersion and external field, cause the soliton coupling, and it is interesting to investigate their mutual influence on the binding process. At last, the dDSG equations contain the sine-Gordon equations as the limit cases.

Thus, we deal with the dispersive double sine-Gordon equation (1dDSG):

$$u_{tt} - u_{xx} - \beta u_{xxx} + \sin(u) + 2h \sin\left(\frac{1}{2}u\right) = 0, \quad (9)$$

and its regularized variant (2dDSG):

$$u_{tt} - u_{xx} - \beta u_{txx} + \sin(u) + 2h \sin\left(\frac{1}{2}u\right) = 0, \quad (10)$$

where, for example, in magnetic applications,  $\phi(x, t) = (1/2) u(x, t)$  denotes the azimuth angle of the magnetization vector in the easy-plane ferromagnet, and  $h$  is a magnetic field applied along the easy plane. When  $h = 0$ , the equations revert to the dispersive and regularized sine-Gordon equations.

Eq. (9) can be derived from the Lagrangian:

$$L = \int_{-\infty}^{\infty} \frac{1}{2} \left\{ u_t^2 - u_x^2 + \beta u_{xx}^2 - 2(1 - \cos(u)) - 8h \left(1 - \cos\left(\frac{u}{2}\right)\right) \right\} dx. \quad (11)$$

The Hamiltonian of the dispersive double sine-Gordon system is given by

$$H = \int_{-\infty}^{\infty} \frac{1}{2} \left\{ u_t^2 + u_x^2 - \beta u_{xx}^2 + 2(1 - \cos(u)) + 8h \left(1 - \cos\left(\frac{u}{2}\right)\right) \right\} dx. \quad (12)$$

Corresponding expressions for the regularized Eq. (10) are obtained by substituting  $u_{tx}^2$  for  $u_{xx}^2$  in Eqs. (11) and (12).

The difference between the dDSG equations results in two types of spectra of linear excitations. Namely, Eqs. (9) and (10) have the dispersion relations  $\omega(k) = \sqrt{1 + h + k^2 - \beta k^4}$  and  $\omega(k) = \sqrt{(1 + h + k^2)/(1 + \beta k^2)}$ , respectively. For the first spectrum, there exists formally the critical wave number  $k_0$  at which  $\omega(k_0) = 0$ . This means that equilibrium state  $u = 0$  is unstable with respect to the short wave length perturbations. Recalling the spectrum of linear excitations of the discrete system, one realizes the artificial origin of this instability. The regularized equation is introduced in order to avoid this instability [33,44] as easily seen from its spectrum. Moreover, such a spectrum shares the main features of Eq. (8), the sine-Gordon model with the nonlocal interaction.

Some important properties of the dSG and dDSG equations can be reproduced by the following model equation which we call as the dispersive double piecewise-linear equation:

$$u_{tt} - u_{xx} - \beta u_{xxx} + f(u) = 0, \quad (13)$$

where force  $f(u)$  is a periodic function of the period  $4\pi$ . On the interval  $[-2\pi, 2\pi]$  it is given by

$$f(u) = \begin{cases} (1+h)(u+2\pi) & -2\pi < u < -u_0 \\ (1-h)u & |u| < u_0 \\ (1+h)(u-2\pi) & u_0 < u < 2\pi. \end{cases} \quad (14)$$

The double quadric potential of the model is presented in Fig. 2. The point, corresponding to the energy maximum, is chosen as  $u_0 = 2 \arccos(h)$  to model the behavior of the double sine-Gordon model. As  $h = 0$  and  $h = 1$ , the equation degenerates into the analogue of the dSG equation with the periods  $2\pi$  and  $4\pi$ , respectively. We show further that this model allows to construct explicitly the expressions for the soliton complexes and their “excited” states.

Other dispersive equations combining the properties of the Klein-Gordon and Boussinesq-type equations are known [34–36], in which topological solitons are shown analytically and numerically to possess an internal structure, and, therefore, they are good candidates for bearing soliton complexes.

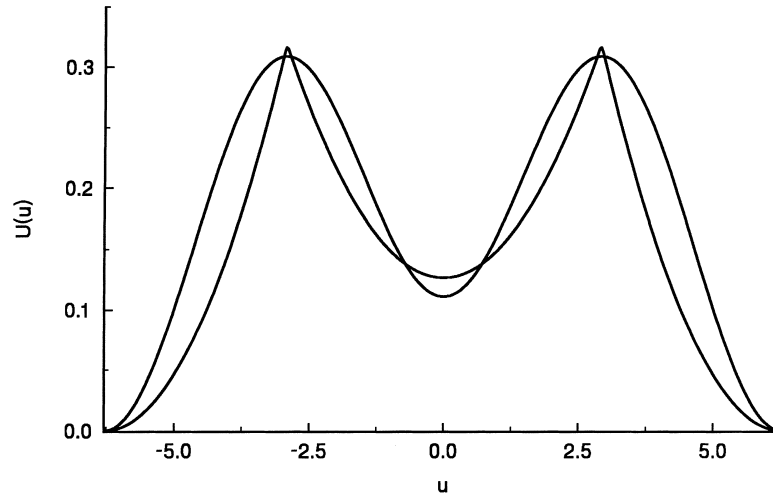


Fig. 2. The potential shapes of the dispersive double sine-Gordon and piecewise-linear models.

### 3. Exact solutions for soliton complexes and their dynamical properties

The exact moving solution of Eq. (4) describes the soliton motion without radiation in a dispersive medium. This radiationless dynamics of a soliton complex in the dSG equation was discussed in detail [30,31]. In this section, we present exact solutions for the dispersive double sine-Gordon equations and discuss the peculiarities of their dynamics. For the concrete definition, we imply the magnetic applications of the equations, where solitons are domain walls and parameter  $h$  is a magnetic field. Therefore, we use further this terminology.

It is known that applying a magnetic field to a ferromagnet leads to coupling  $180^\circ$  domain walls into  $360^\circ$  domain wall. In terms of the usual double sine-Gordon equation ( $\beta = 0$ ) this means the existence of a wobbler solution [41,42] or the  $4\pi$ -kink. Such a static solution does not exist for Eq. (9) if  $\beta \neq 0$  but it holds for Eq. (10):

$$u_w(x) = 4 \arctan \{ \exp(q_w x - R_w) \} + 4 \arctan \{ \exp(q_w x + R_w) \} \quad (15)$$

where  $q_w = \sqrt{1+h}$  and  $\sinh(R_w) = 1/\sqrt{h}$ .

In this work, we are interested in a stationary soliton motion, i.e. consider the solutions of the form  $u(x, t) = u(x - Vt)$ . Then both the Eqs. (9) and (10) are reduced to the ordinary differential equation:

$$u_{zz} + \alpha u_{zzzz} - \sin(u) - 2h \sin\left(\frac{1}{2}u\right) = 0. \quad (16)$$

Here  $z = (x - Vt)/\sqrt{1 - V^2}$ , and parameter  $\alpha$  equals to  $\alpha^{(1)}$  ( $\alpha^{(2)}$ ) for the 1dDSG (2dDSG) equation, where

$$\alpha^{(1)} = \frac{\beta}{(1 - V^2)^2}, \quad \alpha^{(2)} = \frac{\beta V^2}{(1 - V^2)^2}. \quad (17)$$

When  $h = 0$ , Eq. (16) is reduced to the dSG case which was analyzed in [30,31]. Some results of the numerical integration of the reduced equation, as the specific limit of the nonlocal model in Eq. (8), were discussed in [11].

When  $h \neq 0$  we are able to find again the exact solution of Eq. (16)

$$u_{4\pi} = 8 \arctan \left\{ \exp \left( \sqrt{\frac{2}{3} + h} \left( \frac{x - Vt}{\sqrt{1 - V^2}} \right) \right) \right\}. \quad (18)$$

Thus, the 1dDSG and 2dDSG equations have complex solutions in identical form but with different expressions for the velocities

$$V_1(\beta, h) = \pm \sqrt{1 - (2 + 3h)\sqrt{\frac{1}{3}\beta}}. \quad (19)$$

$$V_2(\beta, h) = \pm \left( \sqrt{1 + \frac{1}{3}\beta \left(1 + \frac{3}{2}h\right)^2} - \sqrt{\frac{1}{3}\beta} \left(1 + \frac{3}{2}h\right) \right). \quad (20)$$

These solutions describe complexes consisting of two strongly bound  $2\pi$ -solitons. They differ from the wobbler Eq. (15) by the effective widths, the zero distance between solitons, and the ability to move in the dispersive medium. Since, now velocities are functions of parameter  $h$ , we can change them from the maximum values corresponding to the dSG limit Eqs. (5) and (7) to zero. In Eq. (9), it occurs at the finite critical value  $h_{cr} = \sqrt{(1/3\beta)} - (2/3)$ . For example, in ferromagnets we can control the velocity of a motion of the domain wall complex through the magnetic field.

The next peculiarity of the complex dynamics is revealed, if we evaluate the energies using the Hamiltonian expressions (see Eq. (12)). For the 1dDSG complex we obtain

$$E_1 = 32 \left[ (3\beta)^{-1/4} - \frac{2}{9}(3\beta)^{1/4} \right]. \quad (21)$$

The energy of the soliton complex turns out to be independent of the parameter  $h$ , and, therefore, of the velocity at the fixed  $\beta$ ! This means that at least, by adiabatic variation of the magnetic field, we can vary the form and speed of the complex, conserving its energy. This remarkable property could be used in the energy transfer applications in physical systems described by the 1dDSG equation. However, the property is very much sensitive to spectrum characteristics of the dispersive medium. The regularized Eq. (12) has the velocity and field dependent energy

$$E_2 = 32 \left[ (3\beta V_2^2)^{-1/4} - \frac{2}{9}(3\beta V_2^2)^{1/4} \right], \quad (22)$$

where  $V_2(\beta, h)$  is given by the expression in Eq. (20). More details of comparative analysis of the energy dependences of the complexes are presented in Section 6.

The next natural question is the existence of an exact solution for soliton complexes consisting of more than two solitons. Numerical integrations of the discrete sine-Gordon Eq. (1) and the nonlocal model Eq. (8) reveal such solutions. We are able to find the analytical solution describing the three-soliton complex in the dispersive sine-Gordon equation with sixth spatial derivative [45]:

$$u_{tt} - u_{xx} - \beta u_{xxx} - \gamma u_{xxxxx} + \sin(u) = 0. \quad (23)$$

The equation can be derived from a discrete model, if we take into account the higher order term in an expansion of the type of Eq. (3). For the special choice of the parameters  $\gamma = (3/20)\beta^{3/2}$ , Eq. (23) has the following exact solution:

$$u_{6\pi} = 12 \arctan \left\{ \exp \left( \sqrt{\frac{23}{45}} \frac{(x - V_* t)}{\sqrt{1 - V_*^2}} \right) \right\}, \quad (24)$$

where the velocity takes the form

$$V_* = \pm \sqrt{1 - \frac{23}{30}\sqrt{\beta}}. \quad (25)$$

The exact solution of the form of Eq. (24) exists also in the dispersive triple sine-Gordon equation [45]:

$$u_{tt} - u_{xx} - \beta u_{xxx} - \gamma u_{xxxxx} + \sin(u) + h_1 \sin\left(\frac{1}{3}u\right) + h_2 \sin\left(\frac{2}{3}u\right) = 0, \quad (26)$$

which is the generalization of Eq. (23), and where parameters  $h_1$  and  $h_2$  are arbitrary constants. Under the condition  $\gamma = (3/20)\beta^{3/2}(1 + (1/2h_2))$ , the three-soliton complex is described by the expression

$$u_{6\pi} = 12 \arctan \{ \exp(q_3(x - V_3 t)) \}. \quad (27)$$

The relations between solution parameters are the following:

$$q_3 = \left( \frac{4 + 2h_2}{9\beta} \right)^{1/4}, \quad (28)$$

$$V_3 = \pm \sqrt{1 - \left( \frac{23}{15} + h_1 - \frac{8}{3}h_2 \right) \sqrt{\frac{\beta}{4 + 2h_2}}}. \quad (29)$$

In previous formulas for the presentation of exact solutions, we have used the Lorentz-invariant-like expressions to keep the analogy with those of the Lorentz-invariant equations with  $\beta = 0$ . However, the solutions can be written also in the form like Eq. (27) which is evidently simpler. In any case, at a first glance, the solutions for the soliton complexes and conditions of their existence look like exotic. This raises, at least, three questions. The first of these: how are the solutions sensitive to the variation of the equation parameters? The second, what kind of other solutions exists in the strongly dispersive equations? And the third, what is the mechanism underlying the creation of these bound states? Pondering these questions provides the substance of the next sections.

## 4. Phenomenology of soliton complex

### 4.1. Collective coordinate approach to soliton complex formation

An analytical approach to the description of the soliton complex formation in the dSG equation was proposed in [30,31]. It is based on the use of the collective variable ansatz which is constructed by taking into account the translational and internal degrees of freedom of a soliton as well as interactions between solitons and solitons with radiation

$$u(x, t) = u^{(s)}(x, t; l, X, R) + u^{(r)}(x, t). \quad (30)$$

Here  $u^{(s)}$  is a solitonic part and  $u^{(r)}(x, t)$  is a part of the solution corresponding to radiation. It turns out that the condition of the complex formation of the closely sited solitons can be found from the energy expression of the pair of strongly interacting solitons without taking into account the radiation [31]. Now we use this approximation for the description of the dispersive double sine-Gordon (dDSG) system. So we suppose that the complex dynamics can be considered in the framework of the soliton ansatz

$$u^{(s)} = 4 \arctan \left\{ \exp \left( \frac{x - X}{l} - R \right) \right\} + 4 \arctan \left\{ \exp \left( \frac{x - X}{l} + R \right) \right\}, \quad (31)$$

which is prompted by the forms of the wobbler Eq. (15) and the exact solution in Eq. (18). Here  $l = l(t)$ ,  $X = X(t)$  and  $R = R(t)$  are functions of time. Functions  $l(t)$  and  $X(t)$  describe the changing of the effective width of solitons and their translational motion, respectively. The function  $R = R(t)$  corresponds to the changing separation between solitons, which is defined obviously as  $L = 2lR$ .

Inserting the ansatz into Eqs. (11) and (12), we find the effective Lagrangian and Hamiltonian of two interacting solitons in strongly dispersive media.

$$L_{\text{eff}} = T - U, \quad H_{\text{eff}} = T + U, \quad (32)$$



where  $T$  and  $U$ , the kinetic and potential energies, are given as

$$T = \frac{8}{l} \left\{ \frac{1}{3} l_t^2 \left[ 2R^2 + \left( R^2 + \frac{\pi^2}{4} \right) I_1(R) \right] + (lR_t \tanh(R))^2 I_2(R) + 2RR_t l_t + X_t^2 I_1(R) \right\}, \quad (33)$$

and

$$U = \frac{8}{l} \left\{ I_1(R) + l^2 I_2(R) + 2hl^2 I_0(R) - \frac{\beta}{l^2} \left[ \frac{1}{3} - \tanh^2(R) \left( 1 - I_2(R) \left( 1 + \frac{2}{\sinh^2(2R)} \right) \right) \right] \right\}, \quad (34)$$

where the following notations have been introduced (see also Appendix A):

$$I_0(R) = 2R \coth(R), \quad I_1(R) = 1 + \frac{2R}{\sinh(2R)}, \quad I_2(R) = \coth^2(R) \left( 1 - \frac{2R}{\sinh(2R)} \right).$$

Details of the calculation of explicit expressions for the kinetic and potential energies are presented in Appendix A. The expressions look rather complicated, but they contain a rich information about two-soliton dynamics and interactions. In particular, in the limit cases they describe the exact two-soliton solution of the integrable sine-Gordon equation and the small wobbler oscillations in the usual double sine-Gordon equation (see Appendices B and C).

Since we are interested in the stationary soliton complexes, let us investigate at first the potential energy of two interacting solitons. The energy is drawn in Fig. 3a and b as a function of parameters  $l$  and  $R$  for two different values of the dispersive parameter  $\beta$ . For the sake of symmetry, we show the energy dependence on both positive and negative values of  $R$ , because the ansatz Eq. (31) and the energy are even functions of  $R$ .

Let us examine the character of the dependence of the potential energy on the separation between solitons at the various  $l$  and small  $\beta$ . For the case  $h = 0$  one can see that in the large area near  $l = 1$  the closely placed solitons repulse each other and the energy has a maximum at  $R = 0$ . There exists the critical effective length  $l_{cr}$  at which two local minima appear for the first time at  $R = \infty$ , and then, when  $l$  increases, they quickly diminish their separation. The minima correspond to the equilibrium positions of two solitons in which their mutual repulsion is balanced by the opposite action of the dispersive part. If the magnetic field is present, it gives a strong attractive contribution to the soliton interaction as it is demonstrated in Fig. 3(a), where we take  $\beta = \beta_1 = 1/5$  and  $h = 0.1$ .

In the case when the parameter  $\beta$  is chosen as  $\beta_0 = 3/4$ , one can find that the critical value  $l_{cr}$  corresponds the point where the local maximum in the  $R$ -dependence is changed by a local minimum. This obviously means that in this point the repulsion between solitons is changed into the attraction. This fact can be proved strictly and the critical values can be determined exactly because at small  $R$  the consideration can be performed analytically (see Appendix A for details). Indeed, from the Lagrangian expression Eqs. (A.4)–(A.6) valid for small  $R$ , one derives the following Lagrange equations:

$$\frac{\pi^2}{6} l l_t - \frac{\pi^2}{12} l_t^2 - 1 + X_t^2 + \left( \frac{1}{3} + h \right) l^2 + \frac{\beta}{l^2} = 0, \quad (35)$$

$$\frac{X_t}{l} = p = \text{const.}, \quad (36)$$

$$X_t^2 - 1 + \left( \frac{1}{5} + h \right) l^2 + \frac{7\beta}{5l^3} = 0. \quad (37)$$

The stationary values of  $l_0$  and  $X_t = V_0$  are easily found. They coincide with parameters of the exact solution for the soliton complex, Eqs. (18) and (19), and give the critical values for the parameters at which the soliton attraction arises. It is easy to see, when Eq. (37) is satisfied, that the contribution of terms of  $O(R^2)$  to the “stationary” Lagrangian part Eq. (A.6) equals zero, i.e. at this point it changes sign. That is, for example, for  $h = 0$  and  $V_0 = 0$ , one finds from Eq. (19), the following critical parameter values:  $\beta_0 = 3/4$  and  $l_0(\beta_0) = l_{cr} = \sqrt{3/2} \approx 1.225$ . In fact, at this value  $l_{cr}$ , the potential energy exhibits the flat plot at small  $R$  as it is seen in Fig. 3(b).

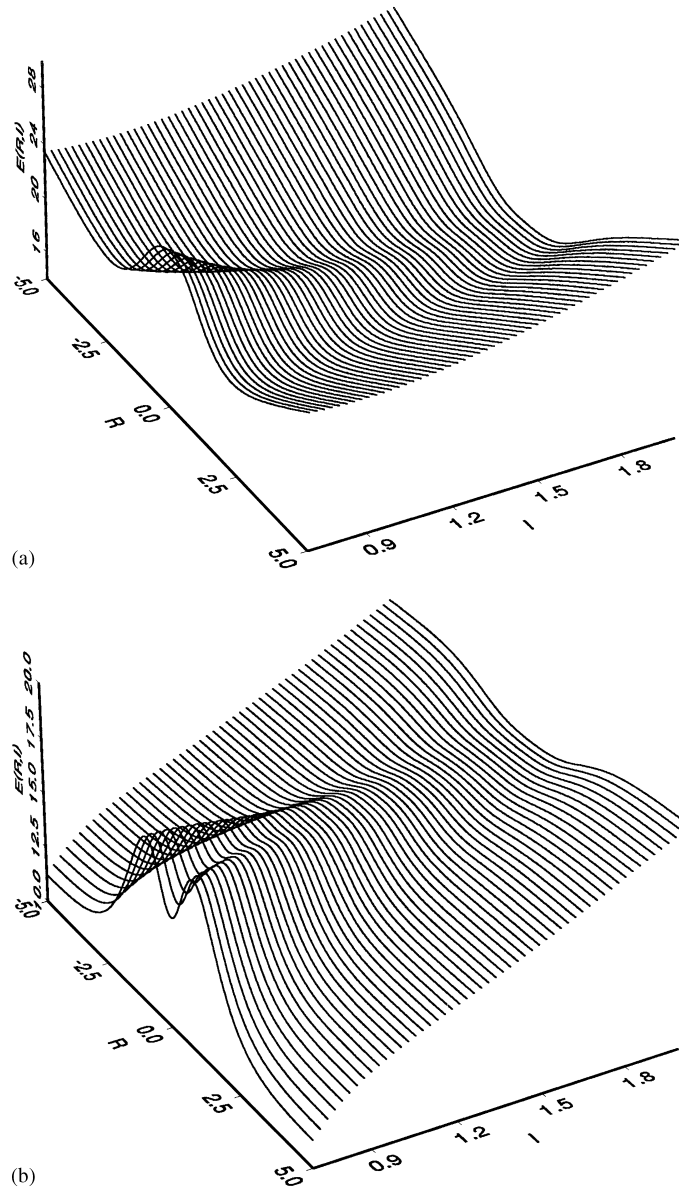


Fig. 3. The potential energy of the dispersive double sine-Gordon system in the variational approach as the function of the effective length and the separation between solitons. Parts (a) and (b) correspond to  $\beta = 1/5$  and  $h = 0.1$ , and to  $\beta = 3/4$  and  $h = 0$ , respectively.

Hence, the possibility of solitons to change their effective length in the dispersive medium leads to changing the very character of interaction between them. Thus, the exact complex solutions correspond to the bound states of solitons coupled in reason of their own attraction.

Stability of the complex with respect to changing its parameters, velocity and effective width, as well as a possibility of its decay can be tested by the numerical simulation of the equations. This work is in progress. Results of [2,11,15] give information about the stability of the complexes in the discrete and nonlocal sine-Gordon model. The complexes manifest themselves as attractors in the soliton dynamics of these dispersive media. Two solitons,

moving with velocities larger than the critical value radiate energy until the velocity reaches its stationary value. Then the formed complex moves radiationlessly. At a velocity smaller than the critical value solitons repulse each other, and the complex decays. As a result, the composite solitons travel from the center to their new equilibrium positions.

This picture is consistent with conclusions following from our direct energy analysis. Further consideration shows that for the first time, equilibrium local minima at  $R = \infty$  appear for the following values of the parameter  $l$ :

$$l(\beta) = \sqrt{\frac{\{[(1/4) + \beta(1+h)]^{1/2} + (1/2)\}}{(1+h)}}. \quad (38)$$

It is easy to see that the parameter  $l_0$  of the exact complex solution also belongs to this dependence. Therefore, at the same moment, when the attraction between solitons arises, the equilibrium local minima disappear and vice-versa. The presence of the local minima points out on a possibility of the existence of another attractor of the wobbler-like type in soliton dynamics. To study the question about the existence of other stationary moving bound states of identical solitons, besides the exact soliton complex, we recur to the ordinary differential Eq. (16).

#### 4.2. Excited states of the soliton complex

The Eq. (16) for stationary states has the asymptotics  $u(z) = A \exp(qz)$  where  $q$  obeys the equation

$$\alpha q^4 + q^2 - 1 - h = 0. \quad (39)$$

Soliton solutions are characterized by the vanishing asymptotics, hence, the corresponding  $q = \kappa$  is supposed to be a real parameter:

$$\kappa = \sqrt{\frac{\{[(1/4) + \alpha(1+h)]^{1/2} - (1/2)\}}{\alpha}}. \quad (40)$$

The exact complex solution is realized when  $\alpha = 3/4$ .

However, Eq. (16), is of the fourth order and has another asymptotics which is oscillating one. In this case  $q = ik$ , where

$$k = \sqrt{\frac{\{[(1/4) + \alpha(1+h)]^{1/2} + (1/2)\}}{\alpha}}. \quad (41)$$

In general, a solution of Eq. (16) includes the solitonic and oscillating parts. This is simply interpreted, because a single moving  $2\pi$ -soliton in the dispersive system usually radiates energy and generates the continuous waves. Thus, in the case of the moving soliton we deal with the self-modulated medium. It is interesting to note that it occurs independently of the stability property of the equilibrium state. In fact, both Eqs. (9) and (10) are reduced to Eq. (16), in spite of different dispersion relations.

When two solitons are present in the system, they create the radiation background, and one soliton moves upon the undulations generated by the other soliton. The situation turns out to be similar to that in the original discrete system [2], where solitons travel upon the Peierls potential. The existence of the purely solitonic stationary excitations under such conditions implies that some interference effect takes place, which leads to cancelling the radiation far from soliton complex. Available theories [25–27] of a soliton binding by radiation field suggest a large separation between composite solitons and a small influence of the dispersion on soliton interactions. It is not correct for closely-sited solitons forming the soliton complex. As a result, an alternative approach to the description of these bound states is required.

To find all possible forms of the soliton complex solutions, one has to solve the nonlinear eigenvalue problem, such as Eq. (16), which is hardly feasible by analytical tools. There are known some attempts to solve similar

problems by variational methods [26,46]. We propose another approach to a searching for soliton complex solutions and demonstrate it by applying to Eq. (16).

So we seek solutions of Eq. (16), imposing the vanishing boundary conditions. Let us reformulate this problem as a self-consistent eigenvalue problem for the linear equation

$$\left[ -\alpha \frac{d^2}{dz^2} + U(z) \right] u_{zz} = 0, \quad (42)$$

where the potential well is

$$U(z) = \frac{\sin(u) + 2h \sin(1/2u)}{u_{zz}} - 1. \quad (43)$$

We know that Eqs. (42) and (43) have, at least, the solution corresponding to the exact soliton complex for  $\alpha = 3/4$ . To investigate the possible existence of other forms of soliton complexes, let us insert the soliton ansatz Eq. (31) with  $R = 0$ :

$$u_0(z) = 8 \arctan \left( \exp \frac{z}{l} \right), \quad (44)$$

into the potential expression Eq. (43). Then the linear Eq. (42) takes the form

$$\left[ -\alpha \frac{d^2}{dz^2} - 1 + l^2 \left( 1 + h - \frac{2}{\cosh^2(z/l)} \right) \right] \psi(z) = 0. \quad (45)$$

After introducing the new coordinate  $y = z/l$ , Eq. (45) can be rewritten as

$$\left[ -\alpha l^{-4} \frac{d^2}{dy^2} + 1 + h - l^{-2} - \frac{2}{\cosh^2(y)} \right] \psi(y) = 0. \quad (46)$$

Solutions of Eq. (46) give us the next approximation for  $u_{zz}$ .

Recalling asymptotics of the eigenvalue problem and Eq. (40), we have to impose the following condition on parameters  $\alpha$  and  $l$ :

$$\alpha l^{-4} = 1 + h - l^{-2} \equiv \epsilon, \quad (47)$$

which must hold for any soliton complex. We have introduced the parameter  $\epsilon$  which takes, evidently, a discrete set of values  $\epsilon_n$ . Thus, finally, we find the equation

$$\left[ -\frac{d^2}{dy^2} + 1 - \frac{2\epsilon_n^{-1}}{\cosh^2(y)} \right] \psi_n(y) = 0. \quad (48)$$

Eigenvalues and eigenfunctions of this type of the equation are well-known [47]. In particular, the equation for parameter  $\epsilon_n$  reads

$$1 = \frac{1}{4} [(1 + 8\epsilon_n^{-1})^{1/2} - 2n - 1]^2. \quad (49)$$

As a matter of fact, the problem is now reduced to the determination of all values of the potential depth in Eq. (48) for which the discrete level equals unity. From Eq. (49) it follows

$$\epsilon_n = \frac{2}{(n+1)(n+2)}, \quad (50)$$

where  $n = 1, 2, 3, \dots$ . Then one obtains for  $l_n$  and  $\alpha_n$

$$l_n^{-2} = h + \sigma_n, \quad (51)$$

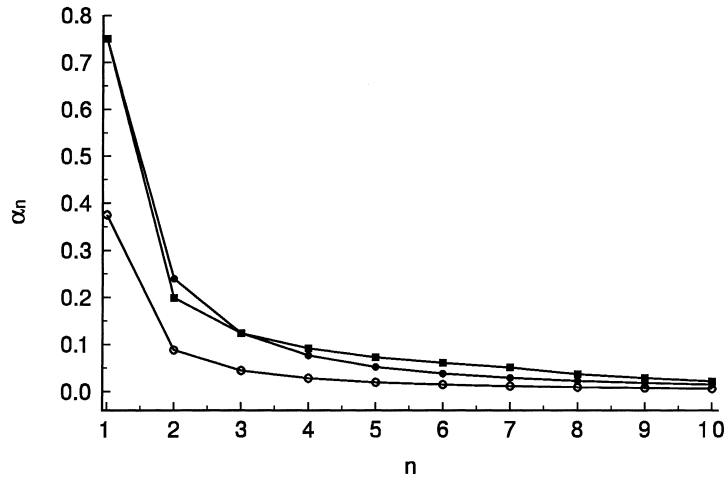


Fig. 4. Eigenvalues  $\alpha_n$  ( $n = 1, \dots, 10$ ) corresponding to the soliton complexes. Solid circles denote results of the phenomenological approach to the stationary dispersive sine-Gordon equation. Open circles correspond to the piecewise-linear model. Solid squares are the eigenvalues obtained by the numerical integration of the dSG equation.

where  $\sigma_n$  equals

$$\sigma_n = \frac{n(n + 3)}{(n + 1)(n + 2)}, \tag{52}$$

and

$$\alpha_n = \alpha_n^0 \left( 1 + \frac{h}{\sigma_n} \right)^{-2}. \tag{53}$$

Here  $\alpha_n^0$  are eigenvalues of the problem for the case  $h = 0$ , i.e. for the dispersive sine-Gordon problem

$$\alpha_n^0 = \frac{2(n + 1)(n + 2)}{[n(n + 3)]^2}. \tag{54}$$

The first 10 eigenvalues  $\alpha_n^0$  are shown as solid circles in Fig. 4 (to show more vividly the dependence on  $n$ , we connect the points by the solid line). The infinite series of discrete values of  $\alpha_n^0$  rapidly diminishes with increasing  $n$ . Only odd values of  $n$  seem to be valid for the problem under the consideration. However, as shown in Fig. 4, the even values of  $n$  also reproduce well enough the exact eigenvalues (solid squares) which we find by numerical integration of Eq. (16) (see Section 6). This is explained by the fact that the next iteration step leads to splitting of levels with  $n > 1$  into two close levels with even and odd eigenfunctions. The level splitting is smaller for higher  $n$ , and the analytical dependence  $\alpha_n^0$  serves as a good approximation of the eigenvalues in the dispersive sine-Gordon case.

Eigenfunctions of Eq. (48) also describe well the changing of the form of  $4\pi$ -soliton when it appears in the excited state. Functions  $\psi_n$ , corresponding to  $u_{zz}$ , vanish exponentially at  $z = \pm\infty$  and exhibit an oscillating behavior at the coordinate origin. The oscillation domain increases for higher  $n$ -values. This oscillation can be interpreted as the radiation locked between two  $2\pi$ -solitons. Thus, the soliton complex can be realized by the infinite series of configurations to be referred to, naturally, as “excited” states.

Formula (53) presents  $\alpha_n$  as functions of the parameter  $h$ . We see that values of  $\alpha_n$  tend quickly to zero at high fields, reflecting correctly the qualitative tendency in the  $h$ -dependent behavior of exact eigenvalues. Quantitative comparison will be done in Section 6.

## 5. Exact soliton complex solutions in double quadric model

Explicit expressions for  $4\pi$ -soliton complex and its “excited” states can be found in the framework of the double piecewise-linear model Eqs. (13) and (14) [45,48]. In this case, stationary moving complexes can be constructed in analytical form, and hence, the corresponding eigenvalue problem for the parameter  $\alpha_n$  can be exactly solved.

In this section, we exhibit principal results of the considerations. For the sake of simplicity, we present main formulas in the limit case  $h = 0$  in reason of their clarity.

As in the dDSG case, after introducing the coordinate  $z$  in the moving reference frame, we derive the equation

$$u_{zz} + \alpha u_{zzzz} - f(u) = 0, \quad (55)$$

where  $f(u)$  is given by the expression Eq. (14). We are interested in odd solutions of Eq. (55),  $u(-z) = -u(z)$ , with limit conditions  $u(\pm\infty) = \pm 2\pi$ . To construct the  $4\pi$ -soliton, it is sufficient to find the solution in the external region ( $z > z_0$ ) and the general odd solution in the internal region ( $|z| < u_0$ ). They look like the followings:

$$u_e(z) = 2\pi - A \exp(-\kappa_2 z), \quad (56)$$

and

$$u_i(z) = B \sin(k_1 z) + C \sinh(\kappa_1 z), \quad (57)$$

In the limit case  $h = 0$ , the exponent  $\kappa_2$  coincides with the parameter  $\kappa_1$  of the solution (57):

$$\kappa_1 = \kappa_2 = \kappa = \pm \sqrt{\frac{[(1/4) + \alpha]^{1/2} - (1/2)}{\alpha}}. \quad (58)$$

The parameter  $k_1$  equals to

$$k_1 = k = \sqrt{\left[ \left( \frac{1}{4} + \alpha \right)^{1/2} + \frac{1}{2} \right] \alpha}. \quad (59)$$

Using the conditions of continuity of the function  $u$  and its first, second and third derivatives in the point  $z_0$ , where  $u = u_0$ , we arrive at the following set of equations:

$$B \sin(kz_0) + C \sinh(\kappa z_0) = 2\pi - A \exp(-\kappa z_0) = u_0, \quad (60)$$

$$kB \cos(kz_0) + \kappa C \cosh(\kappa z_0) = A\kappa \exp(-\kappa z_0), \quad (61)$$

$$-k^2 B \sin(kz_0) + \kappa^2 C \sinh(\kappa z_0) = -A\kappa^2 \exp(-\kappa z_0), \quad (62)$$

$$-k^3 B \cos(kz_0) + \kappa^3 C \cosh(\kappa z_0) = A\kappa^3 \exp(-\kappa z_0). \quad (63)$$

From the system Eqs. (60)–(63), one can find the eigenvalues of the parameter  $\alpha$  and the coefficients  $A$ – $C$ .

It is easy to see from the Eqs. (61) and (63) that  $\cos(kz_0) = 0$ , and hence,  $z_0$  is given by

$$z_0 = \frac{\pi}{k} \left( n - \frac{1}{2} \right) \equiv \frac{\mu_n}{k}. \quad (64)$$

Introducing the parameter  $\lambda(\alpha) = \kappa(\alpha)/k(\alpha)$ , we obtain the following equation

$$\lambda = \exp(-\lambda \mu_n), \quad (65)$$

which determines the eigenvalues  $\lambda_n$  and  $\alpha_n$ . It is clear that this equation has a solution for every  $n$ . Moreover, it is reduced by the substitution  $\Lambda_n = \lambda \mu_n$  to the Lambert’s equation:

$$\Lambda_n \exp(\Lambda_n) = \mu_n. \quad (66)$$

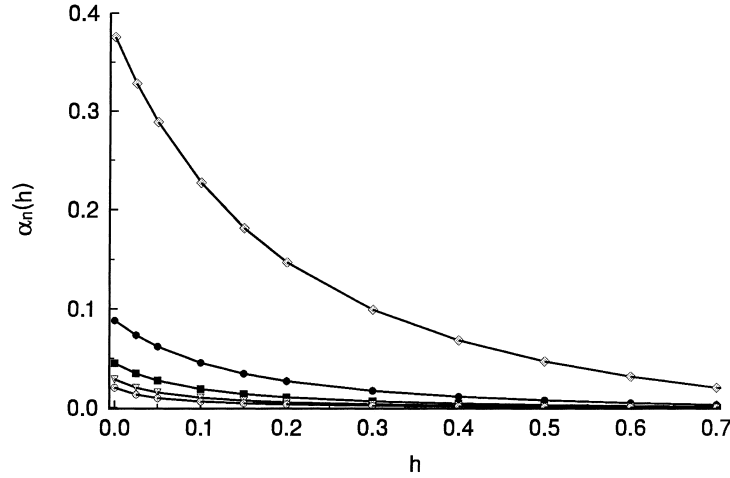


Fig. 5. Dependences  $\alpha_n(h)$  ( $n = 1, \dots, 5$ ) found for the dispersive piecewise-linear model.

Its solution is the Lambert’s function  $\Lambda_n(\alpha) = W(\mu_n)$ . By solving the last equation with respect to  $\alpha$ , we find that there is a infinite series of the  $\alpha_n$ -values for which we can construct the  $4\pi$ -soliton. Coefficients of the solution are expressed through  $k(\alpha_n)$  and  $\kappa(\alpha_n)$  as follows:

$$A = \pi \frac{k}{\kappa}, \tag{67}$$

$$B = (-1)^{n-1} \frac{2\pi\kappa^2}{k^2 + \kappa^2}, \tag{68}$$

and

$$C = \frac{2\pi k\kappa}{k^2 + \kappa^2}. \tag{69}$$

All the 10 eigenvalues  $\alpha_n$  are shown as the open circles in Fig. 4. Note that parameters  $\alpha_n$  diminish with increasing  $n$  in the like manner as in the analytical results Eq. (54).

In the general case  $h \neq 0$ , the equation for  $\alpha_n(h)$  is much more complicated [48] than Eq. (65). The results of its solving are presented in Fig. 5. One can see that  $\alpha_n(h)$  are quickly decaying functions of  $h$ .

Now we discuss properties of the eigenfunctions. Evidently, the soliton complex states can be classified by the integer values of the parameter  $n$ . In the “excited” complexes, the value of  $n$  shows the number of nodes of the oscillating part of the solution. The latter corresponds to the radiation locked between two composite solitons.

The first five eigenfunctions are shown in Fig. 6, where they are numbered from the left to the right by  $n = 1-5$ , respectively. For convenience of observation, we have shifted the centers of complexes in space. With increasing  $n$ , the separation  $L$  between the  $2\pi$ -solitons slowly grows. For large  $n$ ,  $L \sim 2z_0 \simeq 2 \ln(\pi n / \ln(n))$ . At the same time the amplitude of the oscillations decreases, and in the limit  $n \rightarrow \infty$ , the soliton complex is approximated by two well-separated solitons and a linear standing wave between them.

It should also be noted that  $u_z \geq 0$  for all functions. More precisely, at the soliton center  $z_c$  the first derivative reaches its local maximum value for an odd  $n$  and  $u_z = 0$  for the even  $n$ . The latter is possible if the expansion of  $u(z)$  at the point  $z_c$  begins with a term of the order  $(z - z_c)^3$ . Indeed, it is easy to be convinced that the condition  $kB + \kappa C = 0$  is fulfilled for the solution Eqs. (56) with the even  $n$ . The same behavior of the eigenfunctions was found while integrating the dispersive sine-Gordon equation [11]. For the general case  $h \neq 0$ , the condition

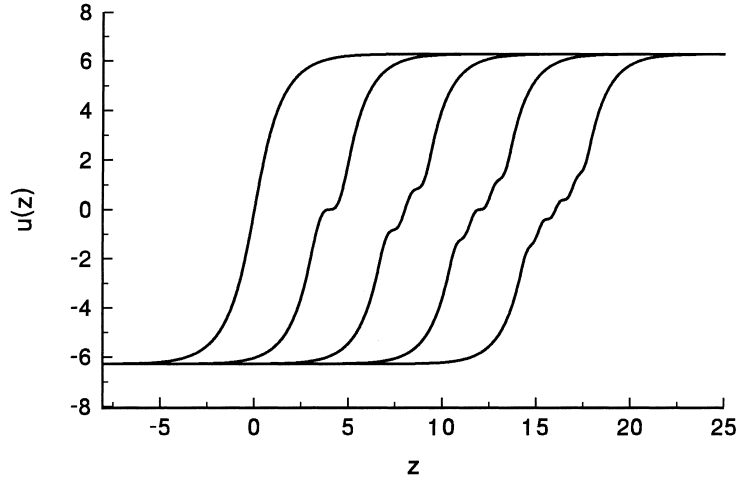


Fig. 6. Five first soliton complex eigenfunctions constructed explicitly in the piecewise-linear model (the case  $h = 0$ ).

$u_z(z_c) = 0$  does not hold in both the piecewise-linear and dDSG models, and the first derivative becomes positive at the soliton center for all  $n$ .

Thus, the simple piecewise-linear model exhibits main peculiarities of soliton complex structure and its stationary dynamics, which turn out to be universal for the dispersive nonlinear models.

## 6. Soliton complexes in the dispersive double sine-Gordon equation

In this section, we present results of the numerical integration of the dDSG equation in the case of the stationary complex motion. We start with Eq. (16) and seek its  $4\pi$ -soliton solution, i.e. impose the limit conditions

$$u(-\infty) = 0, u(\infty) = 4\pi, u_z(\pm\infty) = u_{zz}(\pm\infty) = u_{zzz}(\pm\infty) = 0. \quad (70)$$

However, at first, we propose one more effective approach to solving the equation. It allows us to simplify the integration procedure and formulate some strict assertions about a possibility of different forms of the soliton complexes.

Eq. (16) describes the effective particle dynamics in the four-dimensional space  $\{u, u_z, u_{zz}, u_{zzz}\}$ . It has the first integral

$$I = \alpha[(u_z^2)_{zz} - 3u_{zz}^2] + u_z^2 - 2(1 - \cos(u)) - 8h \left(1 - \cos\left(\frac{1}{2}u\right)\right). \quad (71)$$

By virtue of the limit conditions, (Eq. (70)),  $I = 0$  in the case of the soliton complex solution. In general, the phase space is determined by four variables. However, if we found the solution  $u(z)$ , we could obtain the trajectory in the two-dimensional phase space,  $\{u, u_z\}$  by expressing  $u_z$  as a function of  $u$ .

Let us introduce the function  $u_z^2 = F(u)$  or  $u_z = \sqrt{F}$ . By the differentiation of the definition we get that  $u_{zz} = (1/2)F_u$ . After substitution of these expressions to Eq. (71) we find that the function  $F$  obeys the following second order differential equation:

$$F + \alpha \left(FF_{uu} - \frac{1}{4}F_u^2\right) = 2(1 - \cos(u)) + 8h \left(1 - \cos\left(\frac{1}{2}u\right)\right). \quad (72)$$



From the conditions of Eq. (70) it follows that  $F(u)$  must be periodic with the period  $4\pi$  and  $F_u = 0$  for  $u = 0, 4\pi, \dots$ . It is easy to find the function expansion at  $u = 0$

$$F = \kappa^2 u^2 + O(u^4) + \dots, \tag{73}$$

where  $\kappa$  is given by Eq. (40).

To this end, we are able to perform the numerical integration of the Eq. (72). This is achieved by use of a fourth order Runge–Kutta method. To study the dependence of the soliton complex form on the parameter  $h$ , we begin the integration, at first, for  $h = 0$  (the dSG case). The corresponding results are presented in Figs. 4 and 7 (a, b). In this case, some results can be also verified and compared with those found in the paper [11].

First of all, it is easy to be convinced that there is no  $2\pi$ -kink solution in Eqs. (16) and (72). Next, there are only discrete set of  $\alpha_n$  for which the  $4\pi$ -soliton complexes exist. We show the first 10  $\alpha_n$  as the solid squares in Fig. 4 and see that they are in a good quantitative agreement with the analytical results Eq. (54) (Fig. 4, solid circles). In Fig. 7(a) and (b), the phase portraits are drawn in the plane  $(u, u_z)$ . The main soliton complex trajectory is realized for  $\alpha_1 = 3/4$  and has only one maximum. Other phase portraits differ by the number of maxima. Call this  $n$ . So it

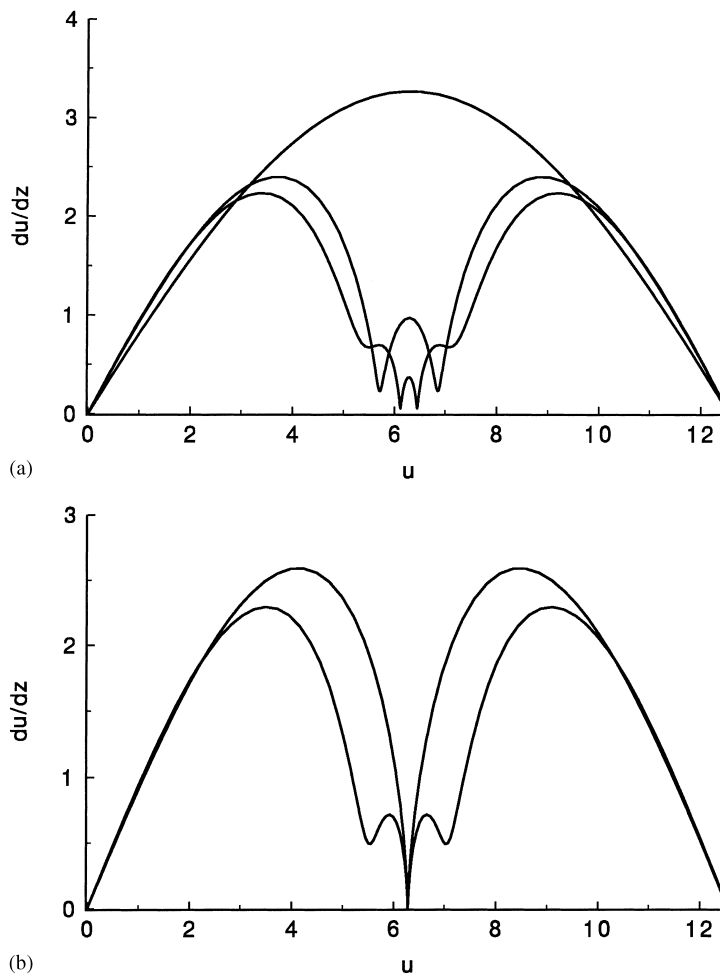


Fig. 7. “Phase portraits” of soliton complex: (a) odd; and (b) even “excited” states in the dSG equation.

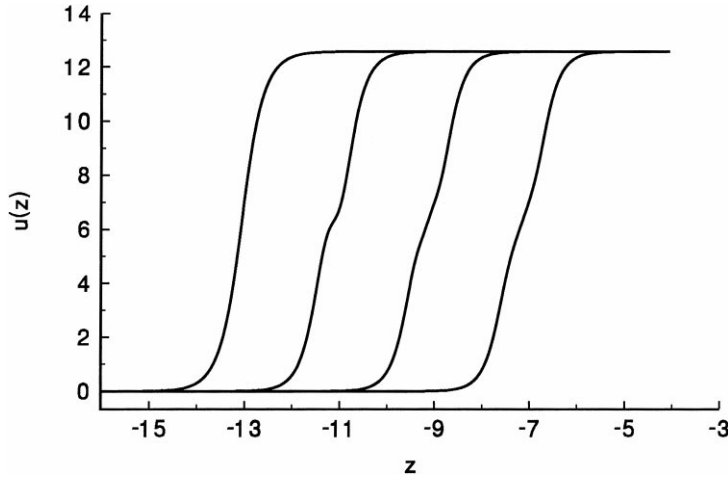


Fig. 8. The soliton complex and its three “excited” states found numerically for the dDSG equation ( $h = 0.1$ ).

is convenient to classify these states by the integer  $n$  using the direct analogy with the results of phenomenological approach and the piecewise-linear model.

In particular, as in the case of the piecewise-linear model, all the odd states (Fig. 7(a)) have the nonzero first derivative at the soliton center where  $u = 2\pi$ . The functions corresponding to even  $n$  (Fig. 7(b)) demonstrate another behavior. Their first derivatives at the point  $u = 2\pi$  behave as  $u_z \sim (u - 2\pi)^{2/3}$ . The reason of the specific dependence is the same as that in the piecewise-linear model, namely, it turns out that  $u(z) - 2\pi \sim (z - z_c)^3$  and hence  $u_z \sim (z - z_c)^2$ . As a result  $u_z = 0$  in the point  $u = 2\pi$  for states with the even  $n$ . However, as we show below, the inclusion of the magnetic field, i.e. taking into account the term with nonzero  $h$ , removes this degeneracy.

Now the functions  $u_n(z)$  for the complex and its “excited” states can be found by numerical integration of the equation  $u_z = \sqrt{F(u)}$ . They virtually have the same shapes as solutions of the piecewise-linear model (c.f. Figs. 6 and 7 of [11]).

When  $h \neq 0$ , the new mechanism of the soliton attraction begins to work. The magnetic field draws together the composite solitons. Results of the numerical solution of the Eq. (72) for this case are shown in Fig. 8. At the same time, one can see that the phase portraits of all the “excited” states become similar with increasing  $n$  even at small  $h$ . It means that in this case the form of the “excited” complex approaches the wobbler solution. The fact is also confirmed by the analysis, performed below, of the dynamical characteristics of the complex and its “excited” states.

Now, we discuss the dependence of  $\alpha_n$  on the parameter  $h$ . For  $n = 1$ , there are exact soliton complex solution of the dDSG equation and the analytical expression for  $\alpha_1(h)$  is given by

$$\alpha_1(h) = \frac{3}{4} \left(1 + \frac{3}{2}h\right)^{-2}. \quad (74)$$

For the “excited” complex states the dependences  $\alpha_n(h)$  are found numerically, and the first five of them are presented in Fig. 9. They are qualitatively similar to those of the piecewise-linear model (Fig. 5) and of analytical results (Eqs. (52)–(54)). Quantitative comparison reveals that with increasing  $h$  numerical eigenvalues  $\alpha_n(h)$  vanish more rapidly than the analytical dependences. However, we believe that the decaying of the functions at high  $h$  is proportional to  $O(1/h^2)$ . We have fitted the data for  $\alpha_n(h)$  by the expressions of the form Eq. (53)

$$\alpha_n = \alpha_n^0 \left(1 + \frac{h}{D_n}\right)^{-2}, \quad (75)$$

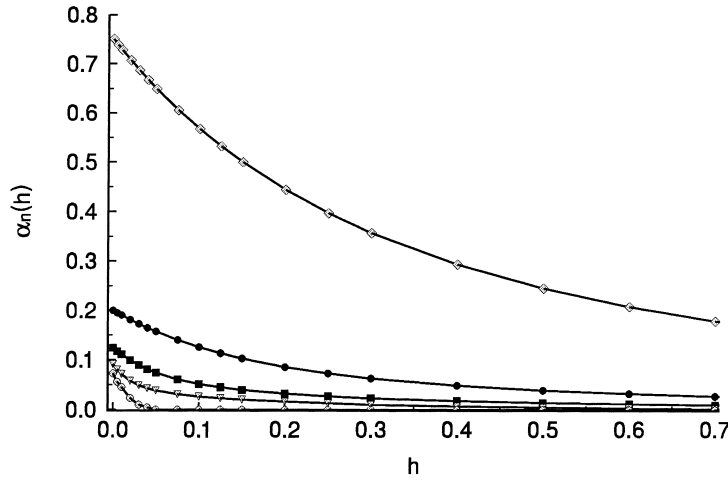


Fig. 9. Numerical results for the eigenvalues  $\alpha_n(h)$  ( $n = 1, \dots, 5$ ) in the dDSG equation.

where  $D_n$  is the only fitting parameter. As a result, we have found a good coincidence between the data and the analytical approximation. For example, in the case  $n = 2$ , the value  $D_2 = 2/5$  provides a deviation less than some percents on the interval  $0 \leq h \leq 2$ . In general,  $D_n$  decreases quickly with the growth of the number  $n$ .

At high  $h$ , the soliton complex and wobbler shapes have to degenerate to the kink of the sine-Gordon equation for the variable  $\phi = u/2$ . Indeed, introducing this new variable one can exactly rewrite Eq. (16) as the following form of the stationary dispersive double sine-Gordon equation:

$$\phi_{\xi\xi} + \tilde{\alpha}\phi_{\xi\xi\xi\xi} - \sin(\phi) - \tilde{h}\sin(\phi)\cos(\phi) = 0, \tag{76}$$

where the following notations are introduced:  $\xi = \sqrt{hz}$ ,  $\tilde{h} = 1/h$ , and  $\tilde{\alpha} = \alpha h$ . When  $\tilde{\alpha} \equiv 0$ , Eq. (76) has the wobbler solution which is reduced to  $2\pi$ -kink in the limit  $\tilde{h} \rightarrow 0$ . The soliton complex and its “excited” states exist at  $\tilde{h} \neq 0$  and  $\tilde{\alpha} \neq 0$ . As it follows from Eq. (75), at high  $h$ , the linear dependence between parameters  $\tilde{\alpha}$  and  $\tilde{h}$  is suggested, i.e.

$$\tilde{\alpha}_n \approx \alpha_n^0 D_n^2 \tilde{h}. \tag{77}$$

In particular, in the case of the exact solution we find from Eq. (74) that

$$\tilde{\alpha}_1(\tilde{h}) = \frac{1}{3}\tilde{h}\left(1 + \frac{2}{3}\tilde{h}\right)^{-2}, \tag{78}$$

and, at small  $\tilde{h}$ , in fact,  $\tilde{\alpha}_1 \approx \tilde{h}/3$ . The eigenfunction  $\phi_1(\xi)$  would be considered as the direct continuation of  $2\pi$ -kink solution of the SG equation for the case  $\tilde{h} \neq 0$ .

The eigenfunctions  $\phi_n(\xi)$  for  $n \geq 2$  behave like the wobbler solution. To understand the proximity of the “excited” states and the wobbler, we apply the iteration procedure (see Section 4.2) to Eq. (76), starting with the wobbler-like ansatz Eq. (31):

$$\phi(\xi) = 2 \arctan \{ \exp(q\xi - R) \} + 2 \arctan \{ \exp(q\xi + R) \}, \tag{79}$$

where now  $q$  and  $R$  are the constant parameters of the solution. The ansatz seems to be advantageous over the previous one, Eq. (44), because it includes two parameters. However, one must keep in mind that already after a first iteration step all parameters get definite values, and the eigenfunction form will be corrected after every step. Here we use the appropriate choice of the ansatz to obtain the analytical estimation for the soliton separation  $R$  in the complex at high field  $h$ .

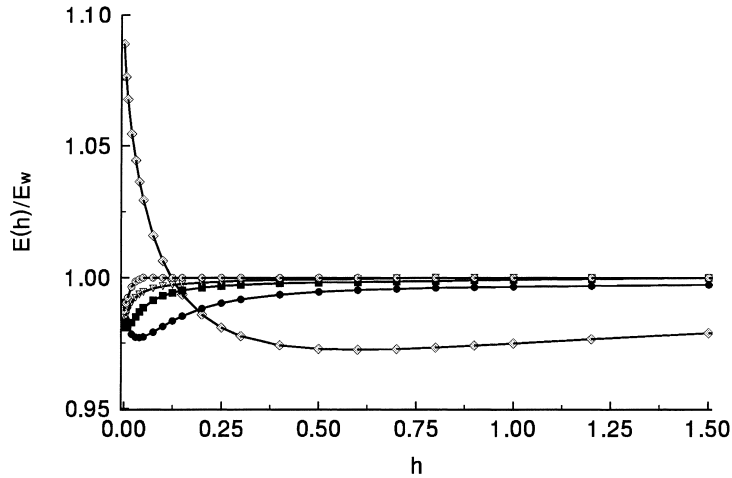


Fig. 10. Energy dependences of the soliton complex and the “excited” states on the parameter  $h$ . The energies are normalized by the wobbler energy  $E_w(h)$ .

Omitting details of the calculation, we derive finally the following equation for the determination of  $\tilde{\alpha}_n(\tilde{h})$ :

$$\left[ -\tilde{\alpha}q^4 \frac{d^2}{d\xi^2} + 1 + \tilde{h} - q^2 - \frac{2(\tilde{h} - \sinh^2(R))}{\cosh^2(\xi) - \sinh^2(R)} \right] \phi_{\xi\xi} = 0. \quad (80)$$

From this equation, in the case of small  $\tilde{h}$ , we find that the quantity  $U_0 = 2(\tilde{h} - \sinh^2(R))/\tilde{\alpha}q^4$  has to be equal to  $(n+1)(n+2)$ . It is easy to see that this relation is in a qualitative agreement with the assumption about the behavior of  $\tilde{\alpha}_n$  (see Eq. (77)). Then we can use Eq. (77) and put  $q \approx 1$  to obtain the following relation for the determination of the parameter  $R$ :

$$\sinh^2(R) \approx \tilde{h}(1 - D_n^2). \quad (81)$$

As it follows from Eq. (81), when  $n$  increases, the fast decay of values  $D_n$  causes the parameter  $R$  rapidly approaches the wobbler value  $R_w$  (see Eq. (15)), even at small  $\tilde{h}$ .

Thus, the smallness of  $\alpha_n(h)$  and its rapidly-vanishing dependence on  $h$  and  $n$ , lead to the degeneracy of the complex “excited” states to the wobbler-like solution.

This fact can be demonstrated perfectly, if one calculates the energies of the soliton complex and its “excited” states as functions of the parameter  $h$  (see Fig. 10). We have normalized these quantities by the energy of the wobbler. As clearly seen from Fig. 10, energies of the “excited” states approach the wobbler energy very quickly with increasing  $h$ , while the exact solution remains well-separated from the wobbler up to high  $h$ .

Another form of the dDSG Eq. (76) can be useful to analyze the hierarchies of the bifurcation values of parameters  $\alpha_m(h)$  for other multisoliton complex solutions. One can note that there is an infinite series of solutions of Eq. (76) for  $\tilde{h} = 0$  and  $\tilde{\alpha} = \tilde{\alpha}_m^0$ , corresponding to the  $4\pi$ -complex for the variable  $\phi$ . When  $\tilde{h} \neq 0$ , these solutions are modified, giving new branches  $\tilde{\alpha}_m(\tilde{h})$ . It is evident that the eigenvalues correspond to the  $8\pi$ -soliton complex and its “excited” states for the variable  $u(z)$  obeying Eq. (16). It is clear that at high  $h$ , the behavior of eigenvalues  $\alpha_m(h)$  is approximated by  $\tilde{\alpha}_m^0/h$ . This is a qualitatively different behavior than that in the case of the  $4\pi$ -complex. At small  $h$ , the parameters  $\alpha_m(h)$  reach their constant bifurcation values  $\alpha_m(0)$ , which can be found from the numerical solution of Eqs. (16) and (17).

It is important to note that a concrete physical system is characterized by the definite value of the dispersive parameter  $\beta$ . Then for the given  $\beta$  and  $h$ , several soliton complex states can exist simultaneously.

In the case of the 1dDSG equation at the fixed  $\beta$ , there exists a discrete set of velocity values  $V_n(h)$  corresponding to the stationary radiationless motion of the complex:

$$V_n(h) = \sqrt{1 - \left(\frac{\beta}{\alpha_n(h)}\right)^{1/2}}, \quad (82)$$

where  $\alpha_n(h)$  are eigenvalues of the nonlinear spectral problem Eq. (16). Every branch has a finite range of the velocity changing from the maximum values  $V_n^0 = \sqrt{1 - (\beta/\alpha_n^0)^{1/2}}$  to zero. It is evident that critical fields  $h_n$  corresponding to static solutions are found from the equation  $\alpha_n(h_n) = \beta$ . The number of possible complex states is finite because the states with  $\alpha_n(h_n) < \beta$  turn out to be forbidden.

The regularized 2dDSG Eq. (10), at any  $\beta$ , has a complete infinite series of soliton complex states with the following allowed velocities:

$$V_n(h) = \sqrt{1 + \frac{\beta}{4\alpha_n(h)}} - \sqrt{\frac{\beta}{4\alpha_n(h)}}. \quad (83)$$

In the case of  $n = 1$ , we have the explicit expression for  $\alpha_1(h)$  (Eq. (74)), and after its substitution to Eq. (83) we come back to Eq. (20). It is clear that the maximum velocity value is realized at  $h = 0$ , when  $\alpha = 3/4$ , it coincides with  $V_1(\beta)$  of Eq. (7). At high  $h$  and hence small  $\alpha_n(h)$  the velocity dependences are simplified to  $V_n(h) \simeq \sqrt{\alpha_n(h)}/\beta$ .

In conclusion, we compare the energies of the complex and two “excited” states as functions of  $h$  for a prescribed  $\beta = 1/12$ . As shown above (see Eq. (21)),  $E_1$  is independent of  $h$ . Energies of the “excited” states turn out to be less than  $E_1$ , therefore, this terminology becomes inadequate when  $\beta$  is fixed. It seems also that the energy arguments about the complex stability do not work in this case. However, it should be noted that the velocities of all the complex states are different, and the exact complex solution possesses the highest velocity. A stability criterion has to be formulated so that the energy comparison is to be performed under the condition of the conservation of the another integral of motion, the momentum. Since for a fixed  $\beta$ , both first integrals are functions of the parameter  $h$  alone, then for a given  $h$ , the momenta are different, so that such a simple comparison becomes impossible. The understanding of the above studied mechanisms of the formation of the topological soliton complexes allows us to believe in their stability in the dispersive conservative systems.

## 7. Summary and discussion

In this section, we discuss some peculiarities of soliton complex dynamics in strongly dispersive models and formulate final general conclusions.

Dispersive nonlinear equations with fourth order spatial derivatives, examples of which have been studied above, are used usually as a first approximation in the description of discrete systems. The present investigation shows that the higher dispersive terms in the nonlinear wave equations effect much more essentially than the small perturbations. The dispersion causes the strong dissipation of energy of the moving  $2\pi$ -soliton, but also makes it possible the creation of the bound soliton complexes consisting of two or more  $2\pi$ -solitons which can move radiationlessly.

The formation of soliton complexes turns out to be the universal phenomenon in nonlinear strongly dispersive media, which are of both theoretical and practical interest.

From the theoretical point of view it is very interesting that the dispersion produces a discrete spectrum for a nonlinear eigenvalue problem for purely solitonic solutions, and its influence is not reduced simply to small changes of the  $2\pi$ -kink shape as it was believed before. Analytical and numerical considerations confirm the absence of the steady  $2\pi$ -kink solution in the above-studied dispersive systems with the fourth and higher order spatial derivatives. The earlier papers (see the review [49] for references) declared the existence of such solutions. Indeed, application

of the perturbation theory, using the smallness of  $\alpha$ , leads to this erroneous conclusion. It seems to be possible to construct formal asymptotic series for the  $2\pi$ -kink solutions of Eqs. (16) and (72). However, summation of all terms of the series results in disappearance of the periodicity of the function  $F(u)$  with period  $2\pi$ , and only solutions with periods  $4\pi, 6\pi, \dots$ , survive.

At the same time these peculiarities of the dispersive effects require some caution when one exploits the variational description of the bound solitons or breather states in dispersive or discrete systems [50].

The numerical finding of the discrete set of periodic eigenfunctions in the Eq. (72) could point to the possibility of a complete integrability of the equation, at least, in some special cases. However, this question is still open.

The existence of soliton complexes consisting of three or more solitons is not discussed in detail in the given paper. The above analysis of the mechanisms and conditions of the creation of the complexes is applicable to the case of the multisoliton bunching. The presence of exact solutions in Eqs. (24) and (27) may be used as a basis for further studies in this direction. One is easily convinced that the exact solutions found are not exotic. Specific relations between constants  $\beta$  and  $\gamma$  of Eqs. (23) and (26) are required only as conditions of the existence of the solution in a very simple analytical form. If we slightly alter the value of the parameter  $\gamma$ , the solution still exists, just as before. Numerical integration of the dSG equation in the stationary case [11] revealed such a solution for  $\gamma = 0$ . Hence the multicomplex solutions are stable with respect to changing the dispersive parameters.

The important question of the stability of the soliton complexes in the framework of the theory of partial differential equations has remained beyond the scope of the paper. Work in this direction is in progress. However, previous studies of the problem in the discrete and nonlocal sine-Gordon models suggest a positive answer to this question.

From the experimental point of view, the soliton complexes may be very attractive for the use in energy and information transfer processes. In particular, the discrete arrays of Josephson junctions can be described by one-dimensional dispersive sine-Gordon models [30,31,50]. The fluxons, the  $2\pi$ -kinks, in these systems could form the soliton complexes with specific properties discussed above. Therefore, Josephson junction arrays may be considered as suitable candidates for real experiments, in which the soliton complexes would manifest themselves [15]. Other examples of appropriate physical systems are the low-dimensional ferro- and antiferro-magnets. More definitely, it may be an one-dimensional biaxial ferromagnet with strong easy-plane anisotropy and a magnetic field applied along the plane. This system is described by the double sine-Gordon equation, and taking into account the discreteness effects duly leads to the dispersive model (Eq. (9)). Two-dimensional antiferromagnets with the weak interplane exchange can be also treated in the framework of the models analyzed in previous sections. The nonlinear dynamics of dislocations is another potential field of application of the obtained results [1,2]. Thus, there are many experimental possibilities for the observation of the occurrence of stationary soliton complex dynamics in dispersive media.

In conclusion, we note that the present investigation shows that purely soliton complexes can be realized in dispersive media which are modulationally unstable from the very beginning, as well as in systems where the  $2\pi$ -kink motion produces the medium modulation due to the radiation accompanying the soliton movement. In both situations nonstationary radiative dynamics of two solitons can result in the final formation of the purely solitonic complex with cancelling the oscillations on soliton wakes, what can be considered as a specific interference effect. However, discussion of the nonstationary soliton complex dynamics as well as the problem of their dynamics in dissipative systems is the subject matter of following publications.

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## Appendix A. Effective Lagrangian and Hamiltonian

After inserting the ansatz Eq. (31) into the Lagrangian Eq. (11) and the Hamiltonian Eq. (12) we perform all needed integrations using the following values of integrals:

$$I_0(R) = \int_{-\infty}^{\infty} dy \frac{\cosh^2(R)}{\cosh^2(R) + \sinh^2(y)} = 2R \coth(R)$$

$$I_1(R) = \int_{-\infty}^{\infty} dy \left( \frac{\cosh(R) \cosh(y)}{\cosh^2(R) + \sinh^2(y)} \right)^2 = 1 + \frac{2R}{\sinh(2R)}$$

$$I_2(R) = \int_{-\infty}^{\infty} dy \left( \frac{\cosh(R) \sinh(y)}{\cosh^2(R) + \sinh^2(y)} \right)^2 = \coth^2(R) \left( 1 - \frac{2R}{\sinh(2R)} \right)$$

$$I_3(R) = \int_{-\infty}^{\infty} dy \left( \frac{\cosh(R)y}{\cosh^2(R) + \sinh^2(y)} \right)^2 = \frac{1}{3} \left( R^2 + \frac{\pi^2}{4} \right) I_0(R)$$

$$I_4(R) = \int_{-\infty}^{\infty} dy \left( \frac{y \sinh(y) \cosh(R)}{\cosh^2(R) + \sinh^2(y)} \right)^2 = \frac{1}{3} [2R^2 \coth^2(R) + \left( R^2 + \frac{\pi^2}{4} \right) I_2(R)].$$

As a result expressions for the kinetic and potential energies of the two-soliton system can be written as

$$T = \frac{8}{l} \left\{ \frac{1}{3} l_t^2 [2R^2 + \left( R^2 + \frac{\pi^2}{4} \right) I_1(R)] + (lR_t \tanh(R))^2 I_2(R) + 2RR_t l_t + X_t^2 I_1(R) \right\}, \quad (\text{A.1})$$

and

$$U = \frac{8}{l} \left\{ I_1(R) + l^2 I_2(R) + 2hl^2 I_0(R) - \frac{\beta}{l^2} \left[ \frac{1}{3} - \tanh^2(R) \left( 1 - I_2(R) \left( 1 + \frac{2}{\sinh^2(2R)} \right) \right) \right] \right\}. \quad (\text{A.2})$$

Then the effective Lagrangian and Hamiltonian are expressed as usual

$$L_{\text{eff}} = T - U, \quad H_{\text{eff}} = T + U. \quad (\text{A.3})$$

The Lagrangian function with accuracy to order  $O(R^2)$  can be written as

$$L = L_0 + R^2 [T_1 - U_1(q, X_t^2)], \quad (\text{A.4})$$

where the zero order Lagrangian is the following one:

$$L_0 = \frac{16}{l} \left\{ \frac{\pi^2}{12} l_t^2 - 1 + X_t^2 - \left( \frac{1}{3} + h \right) l^2 + \frac{\beta}{3l^2} \right\}, \quad (\text{A.5})$$

and  $T_1$  contains all terms proportional to the time derivatives of function  $l$  and  $U_1$  is given by

$$U_1(q, X_t^2) = \frac{16}{3l} \left\{ X_t^2 - 1 + \frac{l^2}{5} + \frac{7\beta}{5l^2} \right\}. \quad (\text{A.6})$$

From Eq. (A.4), we see that one of the Lagrange equation takes the form  $\partial L / \partial R = 0$ . It is transformed into  $R U_1(q, X_t^2) = 0$  for stationary states.

### Appendix B. Integrable sine-Gordon limit: exact two-soliton solution

If we put  $h = 0$ ,  $\beta = 0$ ,  $l = \gamma^{-1}(v) \equiv \sqrt{1-v^2} = \text{const.}$ , and  $X_t = 0$ , then we reduce the problem to the consideration of the sine-Gordon system in the reference frame moving with mass center. In this case, the Hamiltonian  $H_{\text{eff}}$  is simplified to  $H_{\text{SG}}$

$$H_{\text{SG}} = \frac{8}{\sqrt{1-v^2}} \{ (1 + (\tanh(R)R_t)^2) I_2(R)\gamma^{-2} + I_1(R) \}. \quad (\text{B.1})$$

It is easy to be convinced that the function  $R(t)$

$$R(t) = \ln \left[ \frac{1}{v} \cosh(\gamma vt) + \sqrt{\left( \frac{1}{v} \cosh(\gamma vt) \right)^2 - 1} \right] \quad (\text{B.2})$$

is the exact solution of the effective Hamiltonian equations. The substitution of  $R(t)$  into the ansatz (Eq. (31)) and  $H_{\text{SG}}$  provides the exact two-soliton solution of the integrable SG equation and corresponding value for the energy  $E_0 = 16\gamma$ .

### Appendix C. Double sine-Gordon limit: small wobbler oscillations

Suppose again that  $\beta = 0$  but let the parameter  $l$  be the function of  $R$  of the form  $l(R) = \tanh(R)$ . Then after substitution of  $l(R)$  into the Lagrangian, we find for the double sine-Gordon system:

$$L_{\text{DSE}} = 8R_t^2 \tanh(R) G(R) - \coth(R) - hR. \quad (\text{C.1})$$

The last two terms in Eq. (C.1) correspond to the potential energy, and the function  $G(R)$  is introduced as

$$G(R) = I_1(R) + \frac{4}{3 \sinh^2(2R)} \left[ 2R^2 + I_1(R) \left( R^2 + \frac{\pi^2}{4} \right) \right]. \quad (\text{C.2})$$

The stationary solution is found from the condition of minimum of potential energy. One determines the equilibrium value of the parameter  $R$  from the equation  $\sinh(R_w) = 1/\sqrt{h}$ . Naturally, the ansatz coincides with the exact wobbler solution Eq. (15).

Small oscillations near the equilibrium position  $r(t) = R(t) - R_w$  are described by a linear equation which is found from the Lagrangian by the usual way.

$$r_{tt} + \Omega^2 r = 0. \quad (\text{C.3})$$

The frequency is given as

$$\Omega^2(h) = 2h \coth^2 \frac{R_w(h)}{G(R_w(h))}. \quad (\text{C.4})$$

This field dependence for the frequency of the internal oscillations is seemed to be the best analytical approximation of this function for the moment (see [51] and references therein for the comparison).

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