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Nonlinear periodic waves solutions of the nonlinear self-dual network equations

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The new classes of periodic solutions of nonlinear self-dual network equations describing the breather and soliton lattices, expressed in terms of the Jacobi elliptic functions have been obtained. The dependences of the frequencies on energy have been found. Numerical simulations of soliton lattice demonstrate their stability in the ideal lattice and the breather lattice instability in the dissipative lattice. However, the lifetime of such structures in the dissipative lattice can be extended through the application of ac driving terms. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4870649]

I. INTRODUCTION

The models of lattice systems which are described by the integrable differential or differential-difference nonlinear equations are of significant importance for the theoretical and mathematical physics.[1] These equations have the exact solutions allowing to investigate analytically the nonlinear dynamics of the lattice system and make it possible to obtain the explicit expressions for the main physical characteristics of the excitations such as energy, momentum, etc. The most known integrable lattices are the Toda lattice[2] and the system of Ablowitz-Ladik[3] which is the integrable analog of the discrete nonlinear Schrödinger equation.

In 1973 Hirota[4] considered the transmission line with the nonlinear capacitances and inductances (Fig. 1). Transmission line is the system connecting the source and receiver of the signal. Transmission line is, e.g., the pair of wires with length l extended along the certain axis. In cases of practical interest the transverse dimension of the line $d \ll l$ is much smaller than its length. There are a lot of types of the transmission lines—coaxial, microstrip, and other. Coaxial transmission line consists of two coaxial cylindrical metal conductors separated by a dielectric layer. Typically, coaxial cables are used for the transmission of small-power signals in the frequency range from DC to tens of gigahertz. Microstrip transmission line is a strip of highly conductive metal, disposed over the conductive base and separated by a layer of high-quality dielectric. The stripline transmission lines are used in the integrated microwave devices. The great advantage of these lines is the ability to manufacture them using the technology of printed circuits. In this work we consider the transmission lines with the constant transverse configuration along the length. The equivalent circuit of the transmission line can be represented as the series connection of the LC-filters (Fig. 1).

If the geometrical sizes of the system in which the electromagnetic wave propagates are much smaller than the wavelength, the processes can be described in terms of voltage and current strength, instead of the electric and magnetic field strengths (quasi-stationary electric circuit). Hirota[4] proposed the exactly integrable system of the nonlinear self-dual network equations (SDNE) for currents strengths I_n and voltages V_n describing the transmission line with the nonlinear capacitances and

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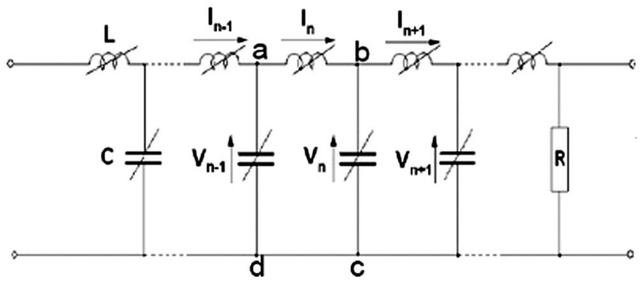

FIG. 1. The equivalent circuit of the ideal nonlinear transmission line.

inductances:

$$\begin{cases} \dfrac{d}{dt}\left[C\left(V_n\right)V_n\right] = I_n - I_{n+1}, \\ \dfrac{d}{dt}\left[L\left(I_n\right)I_n\right] = V_{n-1} - V_n. \end{cases} \tag{1}$$

$$C\left(V_n\right) = V_n^{-1}\arctan V_n, \quad L\left(I_n\right) = I_n^{-1}\arctan I_n. \tag{2}$$

Equations (1) with the circuit parameters (2) are the nonlinear telegraph equations, they can be obtained from the first and second Kirchhoff's rules applied to the equivalent circuit of the transmission line.

After the changes in the variables $V_n = \dot{q}_{n+1/2}$ and $I_n = \dot{q}_n$ Hirota obtained from Eqs. (1) and (2) the equation

$$\frac{dq_n}{dt} = \tan\left(q_{n-1/2} - q_{n+1/2}\right). \tag{3}$$

Dot denotes the derivative in time. Differentiation of the left and right sides of Eq. (3) over time yields the equation for the charge of the nth lattice site:

$$\frac{\ddot{q}_n}{1+\dot{q}_n^2} = \tan\left(q_{n-1} - q_n\right) - \tan\left(q_n - q_{n+1}\right). \tag{4}$$

The mechanical analog of the nonlinear transmission line is a one-dimensional anharmonic chain of atoms, for which only the nearest neighbors interaction is taken into account:

$$\frac{\ddot{u}_n}{1+\dot{u}_n^2} = \tan\left(u_{n-1} - u_n\right) - \tan\left(u_n - u_{n+1}\right), \tag{5}$$

where u_n is the displacement of the nth atom in the chain. In the left part of (5) there is a kinetic term similar as for the modified discrete sine-Gordon model.[5] In the right side there are the nonlinear interaction forces between the nearest neighbors. Equation (5) is called the Hirota lattice equation.[6] Equation (5) is equivalent to the discrete modified Korteweg de Vries equation.[6] In the long-wave limit Eq. (5) reduces to the continuous modified Korteweg de Vries equation.[4] Hirota[4] has shown that Eqs. (1), (2), and (5) are exactly integrable and has found their multi-soliton solutions. Ablowitz and Ladik[7] have found the soliton solutions and the conserved quantities of Eqs. (1) and (2). Bogdan[6] obtained the standing and moving breathers solutions for Eq. (5) for the first time. The discrete breather is a spatially localized and periodic in time nonlinear excitation of the discrete medium.[8] Zhou *et al.*[9] have obtained the discrete breather solution of Eq. (5) using the wronskian technique. In Ref. 10 the discrete breather and wobbling kink[11,12] solutions of Eq. (5) have been found using the nonlinear superposition formula. In Ref. 13 the Hamiltonian dynamics and quasi-classical quantization of the discrete kinks and breathers have been investigated. In Ref. 14 the classical energy spectra of the single nonlinear oscillator of the lattice (5) have been obtained.

To study the effect of dissipative processes and the effect of the external voltage source the model of the transmission line in this work has been modified. The nth section of the modified

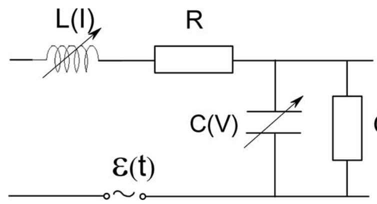

FIG. 2. The nth section of the equivalent circuit of the modified nonlinear transmission line.

nonlinear transmission line is shown in the Fig. 2. The following quantities are introduced: R is the resistance of both current-carrying conductors in the nth lattice site, describing the heat losses in the metal conductors of the transmission line, G is the conductivity of insulation in the nth lattice site, describing the heat loss in the dielectric isolation of the line, and $E(t)$ is the external EMF.

The telegraph equations for the modified transmission line have the form:

$$\begin{cases} \frac{d}{dt}\left[C\left(V_n\right)V_n\right] + GV_n = I_n - I_{n+1}, \\ \frac{d}{dt}\left[L\left(I_n\right)I_n\right] + RI_n = V_{n-1} - V_n + E\left(t\right). \end{cases} \tag{6}$$

For simplicity, we assume $G = 0$. The equation for the current strength can be obtained from Eqs. (6) and (2):

$$\begin{aligned} &\frac{d}{dt}\left(\arctan I_n\right) + RI_n = \tan\left\{\int dt\left[I_{n-1}\left(t\right) - I_n\left(t\right)\right]\right\} - \\ &- \tan\left\{\int dt\left[I_n\left(t\right) - I_{n+1}\left(t\right)\right]\right\} + E\left(t\right). \end{aligned} \tag{7}$$

Introducing the electrical charge for the nth site $I_n = \dot{q}_n$ one can obtain the following equation:

$$\frac{\ddot{q}_n}{1+\dot{q}_n^2} + R\dot{q}_n = \tan\left(q_{n-1} - q_n\right) - \tan\left(q_n - q_{n+1}\right) + E\left(t\right). \tag{8}$$

In the last decade the nonlinear solitonic transmission lines are investigated intensively.[15] A great variety of the electrical solitonic transmission lines and devices, providing the generation, filtering, and amplification of the solitonic pulses have been produced. Ricketts *et al.*[16] have introduced the electronic device generating a periodic stable train of electrical solitons electrical soliton oscillator. This waveform is also known as a cnoidal wave. Kovalev[17] has obtained the cnoidal waves solutions of the sine-Gordon equation. In Ref. 18 the breather lattice solution of the sine-Gordon equation has been investigated. In Ref. 19 the exact periodic solutions of the positive and negative modified Korteweg-de Vries equations have been found.

The mechanical analog of Eq. (8) has the form:

$$\begin{aligned} &\frac{\ddot{u}_n}{1+\dot{u}_n^2} = \tan\left(u_{n-1} - u_n\right) - \\ &- \tan\left(u_n - u_{n+1}\right) - \lambda\dot{u}_n + f\left(t\right). \end{aligned} \tag{9}$$

The third and fourth terms in the right side of Eq. (9) are the friction force and the external variable homogeneous force. The Lagrangian function for Eq. (9) has the form:

$$\begin{aligned} &L = \sum_{n=-\infty}^{+\infty} L_n, \quad L_n = \dot{u}_n \arctan\dot{u}_n - \frac{1}{2}\ln\left(1+\dot{u}_n^2\right) + \\ &+ \frac{1}{2}\ln\cos^2\left(u_n - u_{n-1}\right). \end{aligned} \tag{10}$$

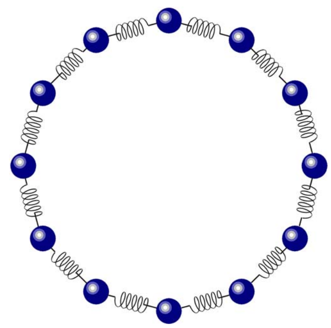

FIG. 3. Periodic boundary conditions.

Equation (5) is exactly integrable, it has the infinite number of integrals of motion, one of them is energy, which can be obtained from the Lagrangian (10):

$$E = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \left[\ln \left(1 + \dot{u}_n^2 \right) - \ln \cos^2 \left(u_n - u_{n-1} \right) \right]. \tag{11}$$

In this work the new solutions of the Hirota lattice equation (5) in the form of the moving and standing nonlinear periodic waves have been obtained. The lattice consisting of the arbitrary but finite number of sites has been considered. Equation (5) is completed with the boundary conditions. The chains with zero fixed and periodic boundary conditions have been investigated. For solutions satisfying the given boundary conditions, the dependence of the oscillation frequency on energy has been found. It was shown that in the linear (small amplitude) limit the obtained solutions reduce to the linear waves and in the essentially nonlinear (large amplitude) case to the breather, soliton-soliton, or soliton-antisoliton sequences. For the nonlinear transmission lines the dissipative processes have been investigated numerically and the mechanism of the breather sequences stabilization in dissipative transmission lines has been suggested.

II. PERIODIC AND ZERO FIXED BOUNDARY CONDITIONS

In this work the finite size transmission lines and equivalent atomic chains with an arbitrary but finite number of sites which are described by (1), (2), and (5) are investigated. In order to consider the influence of the boundaries one has to complete the equation of motion with the boundary conditions. In this work the periodic (Born-von Karman) and zero fixed boundary conditions have been discussed. Periodic boundary conditions correspond to the transmission line (or atomic chain) closed in the ring, though we consider the one-dimensional motion: $u_{N+1}(t) = u_1(t)$ (Fig. 3). If the end of the transmission line is open-looped then the current strength in the boundary node equals to zero ($I_N(t) = 0$), if the end of the transmission line is shorted then the voltage in the boundary equals zero ($V_N(t) = 0$). For Eq. (4) zero fixed boundary conditions correspond to the zero charge in the boundary nodes. In the mechanical model the zero fixed boundary conditions correspond to the zero displacements of the boundary atoms: $u_1(t) = 0$, $u_N(t) = 0$ (Fig. 4).

It is easy to show that differentiation in time of the energy for a finite number of nodes in the sum (11) gives zero for periodic and zero fixed boundary conditions. Thus Eq. (5) for the finite size

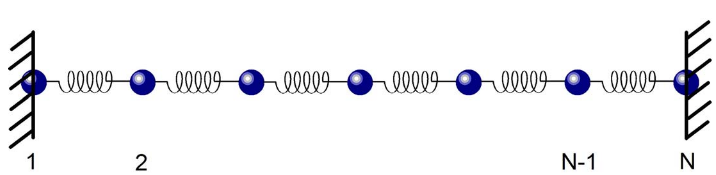

FIG. 4. Zero fixed boundary conditions.

lattice with the periodic and zero fixed boundary conditions also has the energy as the integral of motion.

III. BREATHER LATTICE SOLUTION OF TYPE I

For brevity we will use the following notation:

$$s_1 \equiv \mathrm{sn}\,(a/2,\chi)\,,\, c_1 \equiv \mathrm{cn}\,(a/2,\chi)\,,\, d_1 \equiv \mathrm{dn}\,(a/2,\chi)\,,$$
$$s_2 \equiv \mathrm{sn}\,(p/2,m)\,,\, c_2 \equiv \mathrm{cn}\,(p/2,m)\,,\, d_2 \equiv \mathrm{dn}\,(p/2,m)\,, \tag{12}$$

$$\chi' = \sqrt{1-\chi^2}, \quad m' = \sqrt{1-m^2}, \tag{13}$$

where $\mathrm{sn}\,(\theta,\chi)\,,\,\mathrm{cn}\,(\theta,\chi)\,,\,\mathrm{dn}\,(\theta,\chi)$ are the Jacobi elliptic functions, χ, m are the modules of the elliptic functions, χ', m' are the additional modules of the elliptic functions (see Appendix A for the Jacobi functions properties).

In this paper the new solution of Eq. (5) in the form of a moving nonlinear periodic wave has been obtained for the first time (see Appendix B for details):

$$u_n\,(t) = \arctan\left[A\,\mathrm{sn}\,(an+bt,\chi)\,\mathrm{dn}\,(pn+qt,m)\right], \tag{14}$$

where

$$b = \pm\frac{2s_1d_2}{1-A^2s_1^2d_2^2}, \quad q = \pm\frac{2s_2c_1c_2d_1}{1-A^2s_1^2d_2^2}, \tag{15}$$

$$A = \sqrt{\frac{\chi}{m'}} = \frac{s_2}{s_1c_2}. \tag{16}$$

From expression (16) the conditions on the parameters p, m have been obtained

$$\mathrm{sn}\left(\frac{p}{2},m\right) \leqslant \frac{1}{\sqrt{1+m'}}.$$

The solution (14) we will call the breather lattice solution of type I. The quantities $V_{ph} = b/a$ and $V_g = q/p$ would be considered as the counterparts of the phase and group velocity of the linear wave packet for the nonlinear case. Below the limiting cases of the solutions (14)–(16) are analyzed.

For $m \to 0$ the solution reduces to the form

$$u_n\,(t) = \arctan\left[\sqrt{\chi}\,\mathrm{sn}\,(an+bt,\chi)\right], \tag{17}$$

where

$$b = \pm\frac{2s_1}{1-\chi s_1^2}. \tag{18}$$

For $\chi \to 0$, $m \to 0$ the solution (14)–(16) reduces to the linear wave: $u_n(t) = A\sin\,(an + bt)$ with the amplitude as a small parameter ($A \to 0$). For small values of the atoms displacements Eq. (5) reduces to the well known equation describing the oscillations of the linear 1D crystal. In the harmonic crystal the dependence of the linear wave cyclic frequency b on the quasi-wave number a has the familiar form[20] $b = \pm 2\sin\,(a/2)$.

For $\chi \to 0$, $m \to 1$ the solution (14)–(16) reduces to the moving discrete breather.[6]

$$u_n\,(t) = \arctan\left[\frac{\sinh\,(p/2)}{\sin\,(a/2)}\frac{\sin\,(an+bt)}{\cosh\,(pn+qt)}\right], \tag{19}$$

where

$$b = \pm 2\cosh\,(p/2)\sin\,(a/2)\,,$$
$$q = \pm 2\sinh\,(p/2)\cos\,(a/2)\,. \tag{20}$$

For $\chi \to 1$, $m \to 0$ the solution (14)–(16) reduces to the one-parametric soliton (kink ($a' > 0$) or antikink ($a' < 0$)) : $u_n\,(t) = \arctan\left[\exp\left(a'n+bt\right)\right] - \pi/4$, where $b = \pm 2\sinh\left(a'/2\right)$ and $a' = 2a$.

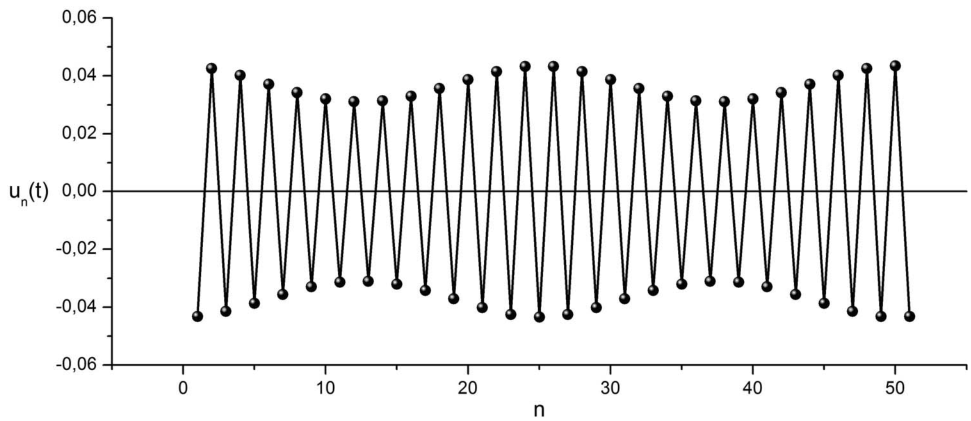

FIG. 5. The graph of the breather lattice solution of type I. $N = 50$, $M = 2$, $\omega = 2.0083$, $m = 0.7$, and $\chi = 0.004$. The boundary conditions are periodic.

Below the nonlinear analogue of the linear standing wave is considered. To obtain the standing wave one has to put $V_g = 0$, i.e., $q = 0$, $\forall p$, then the following condition holds : $\mathrm{cn}\,(a/2, \chi)\,\mathrm{dn}\,(a/2, \chi) = 0$. Since the function $\mathrm{dn}\,(\theta, m)$ never equals to zero: $\mathrm{dn}\,(\theta, \chi) \neq 0,\ \forall\theta$, one can obtain the condition : $\mathrm{cn}\,(a/2, \chi) = 0$, then the parameter a has to take the following values:

$$a = 2\,(2l + 1)\,\mathbb{K}\,(\chi)\,, \quad l = 0, \pm 1, \pm 2, \ldots. \tag{21}$$

Using the periodicity property of the Jacobi elliptic function (A10) the solution describing the nonlinear inhomogeneous antiphase oscillations in the chain of atoms (Figs. 5 and 6) is obtained from Eqs. (14)–(16)

$$u_n\,(t) = (-1)^n \arctan\,[A\mathrm{sn}\,(bt, \chi)\,\mathrm{dn}\,(pn, m)]\,, \tag{22}$$

where

$$b = \pm \frac{2d_2}{1 - A^2 d_2^2}, \tag{23}$$

$$A = \sqrt{\frac{\chi}{m'}} = \frac{s_2}{c_2}. \tag{24}$$

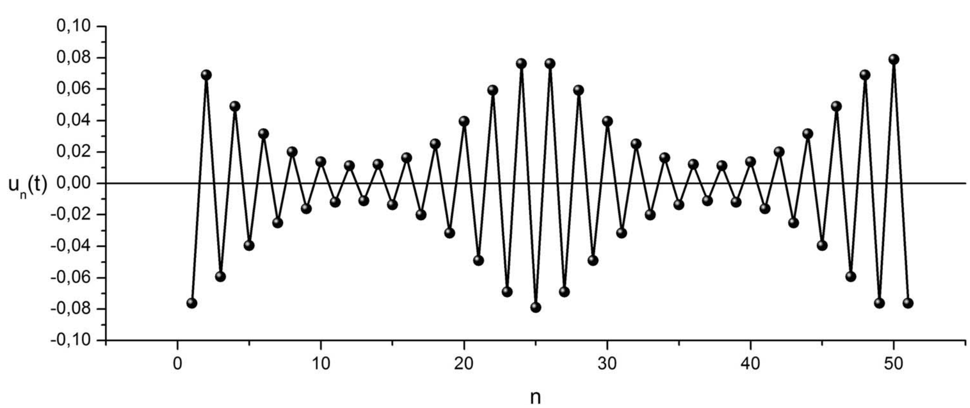

FIG. 6. The graph of the breather lattice solution of type I. $N = 50$, $M = 2$, $\omega = 2.0184$, $m = 0.99$, and $\chi = 0.003$. The boundary conditions are periodic.

It is well known that the period of oscillations is equal $T = 4\mathbb{K}(\chi)/b$, and the cyclic frequency of the nonlinear inhomogeneous antiphase oscillations is

$$\omega = \frac{2\pi}{T} = \frac{\pi}{2\mathbb{K}(\chi)}b. \tag{25}$$

For $m \to 0$ Eqs. (22)–(24) reduce to the solution describing the nonlinear homogeneous antiphase oscillations.

$$u_n(t) = (-1)^n \arctan\left[\sqrt{\chi}\,\mathrm{sn}(bt, \chi)\right], \tag{26}$$

where

$$b = \pm\frac{2}{1-\chi}. \tag{27}$$

The cyclic frequency of the nonlinear homogeneous antiphase oscillations is equal

$$\omega_0 = \frac{\pi}{\mathbb{K}(\chi)}\frac{1}{(1-\chi)}. \tag{28}$$

For $\chi \to 0$, $m \to 1$ Eqs. (22)–(24) reduce to the solution describing the standing discrete breather:

$$u_n(t) = (-1)^n \arctan\left[\sinh(p/2)\frac{\sin bt}{\cosh pn}\right], \tag{29}$$

where

$$b = \pm 2\cosh(p/2). \tag{30}$$

Equation (14) and all its limiting cases represent the solution of Eq. (5) for the infinite 1D lattice. For the finite length chain one has to consider the boundary conditions. In this case the additional conditions for the solution parameters are imposed.

Since the function $\mathrm{dn}(\theta, m) \neq 0,\ \forall\theta$ zero fixed boundary conditions are not satisfied for the solution (22) and for its limiting cases.

Suppose that the length of the chain contains the integer number M of the real spatial periods of the function $\mathrm{dn}(pn, m)$. Then $pN = 2\mathbb{K}(m)M$. Here the well known periodicity property of the Jacobi elliptic function (A12) is used. Periodic boundary conditions ($u_{n+N}(t) = u_n(t)$) are satisfied if

$$p = \frac{2\mathbb{K}(m)M}{N}, \quad M = 1, 2, 3, \ldots. \tag{31}$$

Using the condition imposed to the parameters p, m, the following condition can be obtained:

$$\mathrm{sn}\left(\frac{\mathbb{K}(m)M}{N}, m\right) \leqslant \frac{1}{\sqrt{1+m'}}, \quad M = 1, 2, 3, \ldots.$$

From (22) and (31) it is seen that if the periodic boundary conditions are considered then N must be even. If N/M is even then the number of the solution spatial periods that fit the length of the chain equals M. If N/M is odd then the number of the solution spatial periods that fit the length of the chain equals $M/2$.

Below the dependences of the cyclic frequency on energy for the homogeneous and inhomogeneous oscillations are obtained in the case when the periodic boundary conditions have been used.

The cyclic frequency of the nonlinear homogeneous antiphase oscillations is defined by (28). The energy of the homogeneous oscillations in the chain with N sites is equal

$$E_0 = N\ln\frac{1+\chi}{1-\chi}. \tag{32}$$

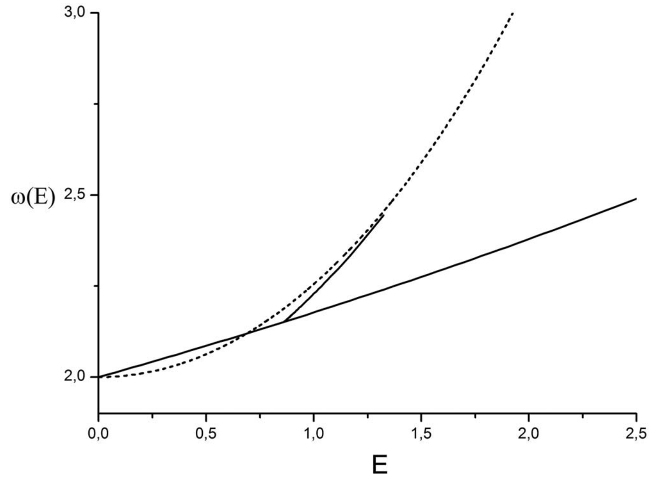

FIG. 7. The dependence of frequency on energy $\omega = \omega(E)$ for the homogeneous and inhomogeneous oscillations of the chain with the number of sites $N = 6$ and $M = 1$. The boundary conditions are periodic. Dashed line shows the dependence of discrete breather frequency on energy for the infinite chain.

The frequency of the nonlinear inhomogeneous antiphase oscillations is defined by (25) and (23). The energy of the inhomogeneous oscillations in the chain with N sites is equal

$$E = \frac{1}{2}\sum_{n=1}^{N} \ln\left(1 + A^2 b^2 \mathrm{dn}^2\left(pn, m\right)\right). \tag{33}$$

Bogdan and Maugin[6] have obtained the expressions for the frequency and energy of the standing discrete breather of Eq. (5):

$$\omega_b = 2\cosh\left(\kappa/2\right), \quad E_b = \kappa. \tag{34}$$

In Fig. 7 the dependence of frequency on energy for the homogeneous and inhomogeneous oscillations of the chain with the number of sites $N = 6$ with the periodic boundary conditions is shown. The number of spatial periods that fit the length of the chain $M = 1$. Dashed line shows the dependence of discrete breather frequency on energy for the infinite chain.

The branch corresponding to the inhomogeneous oscillation is detached from the branch corresponding to the homogeneous oscillation by the bifurcation way and then approaches the breather branch (dashed line).

From Fig. 7 it is seen that the inhomogeneous oscillation is energetically more favorable than the corresponding homogeneous oscillation of the same frequency.

To examine the dynamical stability of the obtained solutions the direct numerical simulations have been used. The time integration has been performed by means of a 7th order Runge-Kutta scheme. The space-time evolution of the breather lattice solution of the type I for the ideal (Fig. 8) and dissipative (Fig. 9) lattice with $\lambda = 0.01$ has been investigated. The periodic boundary conditions have been used. Simulations demonstrate the metastability of the solution in the ideal lattice with long life time and instability in the dissipative lattice.

IV. BREATHER LATTICE SOLUTION OF TYPE II

The another type of the exact periodic solution of Eq. (5) can be found as follows:

$$u_n\left(t\right) = \arctan\left[A\mathrm{cn}\left(an + bt, \chi\right)\mathrm{cn}\left(pn + qt, m\right)\right], \tag{35}$$

where

$$b = \pm\frac{2s_1c_2d_1\left(1 + A^2\right)}{1 + A^2c_1^2c_2^2}, q = \pm\frac{2s_2c_1d_2\left(1 + A^2\right)}{1 + A^2c_1^2c_2^2}, \tag{36}$$

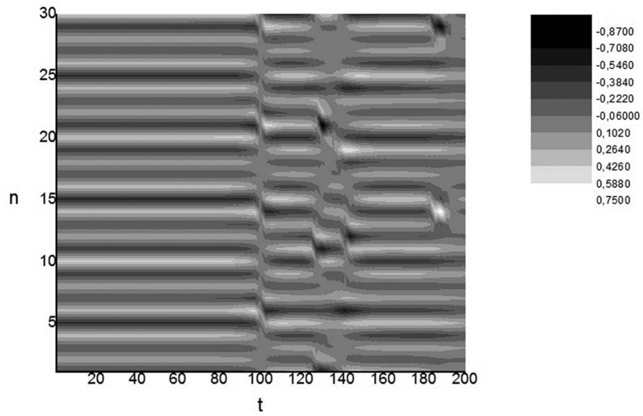

FIG. 8. A space time contour plot $u_n(t)$. Initial condition is the breather lattice solution of type I. $N = 30$, $M = 6$, $m = 0.90$, and $F = 0$. Time is measured in the periods of the oscillation (22). Boundary conditions are periodic. The dynamics of the ideal lattice. $\lambda = 0$.

$$A = \sqrt{\frac{\chi m}{\chi' m'}} = \sqrt{\frac{d_2^2 - d_1^2}{c_1^2 d_2^2 - c_2^2 d_1^2}}. \tag{37}$$

Solution (35) represents the nonlinear periodic wave. From (37) it is seen that the following condition has to be hold:

$$\frac{d_2^2 - d_1^2}{c_1^2 d_2^2 - c_2^2 d_1^2} \geqslant 0.$$

When considering the limiting cases of solutions and their parameters, expressed in terms of the Jacobi elliptic functions the well known relations for the elliptic functions have been used (see Appendix A).

For $\chi \to 0$, $m \to 0$ the solution (35) reduces to the linear superposition of two linear waves

$$\begin{aligned} u_n(t) &= A\cos(an + bt)\cos(pn + qt) = \\ &= \frac{A}{2}\cos\left[(a+p)n + (b+q)t\right] + \frac{A}{2}\cos\left[(a-p)n + (b-q)t\right], \end{aligned} \tag{38}$$

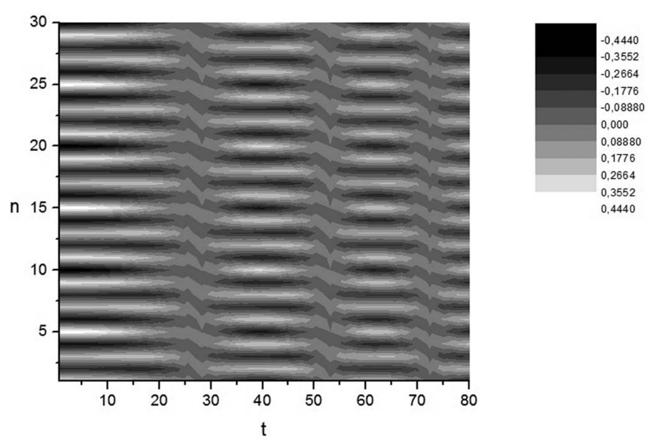

FIG. 9. A space time contour plot $u_n(t)$. Initial condition is the breather lattice solution of type I. $N = 30$, $M = 6$, $m = 0.90$, and $F = 0$. Time is measured in the periods of the oscillation (22). Boundary conditions are periodic. The dynamics of the lattice with dissipation. $\lambda = 0.01$.

where

$$b = \pm 2 \sin(a/2) \cos(p/2),$$
$$q = \pm 2 \cos(a/2) \sin(p/2), \tag{39}$$

and the amplitude of the linear waves is a small parameter: $A \to 0$.

For $\chi \to 0$, $m \to 1$ solution (35) reduces to the traveling discrete breather solution (19), (20). From the symmetry of the solution (35)–(37) it is obvious that for $\chi \to 1$, $m \to 0$ solution (35) also reduces to the travelling discrete breather.

From (35) the limiting case describing the nonlinear oscillations is obtained if one sets $b = 0$, $\forall a$, and hence $\mathrm{cn}(p/2, m) = 0$. Then it follows that

$$p = 2(2l+1)\mathbb{K}(m), \quad l = 0, \pm 1, \pm 2, \ldots. \tag{40}$$

Using the periodicity property of the Jacobi elliptic function (A11) from Eq. (35) the following expression has been found:

$$u_n(t) = (-1)^n \arctan[A \mathrm{cn}(an, \chi)\mathrm{cn}(qt, m)], \tag{41}$$

$$q = \pm 2m' \left(1 + A^2\right) c_1, \tag{42}$$

$$A = \sqrt{\frac{\chi m}{\chi' m'}} = \frac{\sqrt{\chi^2 s_1^2 - m^2}}{c_1 m'}. \tag{43}$$

From expression (43) the conditions on the parameters a, χ, m have been obtained $m/\chi \leqslant |\mathrm{sn}(a/2, \chi)| \leqslant 1$. Therefore,

$$m \leqslant \chi, \tag{44}$$

$$m \leqslant \chi\, |\mathrm{sn}(a/2, \chi)|. \tag{45}$$

Obviously, the period of oscillations is equal $T = 4\mathbb{K}(m)/q$ and the cyclic frequency of oscillations is related to the period in the standard way:

$$\omega = \frac{2\pi}{T} = \frac{\pi}{2\mathbb{K}(m)} q. \tag{46}$$

For $\chi \to 0$, $m \to 0$ solution (41) reduces to the standing linear wave $u_n(t) = (-1)^n A \cos an \cos qt$, where $q = \pm 2\cos(a/2)$. The amplitude of the standing linear wave $(-1)^n A \cos an$ is a small parameter since $A \to 0$.

For $\chi \to 1$, $m \to 0$ solution (41) reduces to the standing discrete breather. Zero fixed boundary conditions are satisfied if $\mathrm{cn}(a, \chi) = \mathrm{cn}(aN, \chi) = 0$. Then

$$a = \mathbb{K}(\chi)(2l_1 + 1), \quad l_1 = 0, 1, 2, \ldots, \tag{47}$$

$$aN = \mathbb{K}(\chi)(2l_2 + 1), \quad l_2 = 1, 2, 3, \ldots. \tag{48}$$

For simplicity we set

$$a = \mathbb{K}(\chi). \tag{49}$$

From (48) and (49) it follows that in order to satisfy zero fixed boundary conditions the number of sites must be odd (Fig. 10).

Suppose the length of the chain contains the integer number $y = 1, 2, \ldots$ of real spatial half-periods of the function $\mathrm{cn}(an, \chi)$. Then $a(N-1) = 2y\mathbb{K}(\chi)$. From expression (49) it follows that

$$y = \frac{N-1}{2}. \tag{50}$$

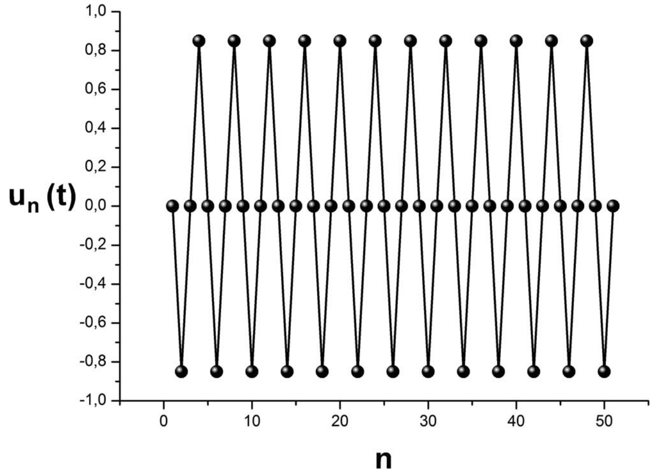

FIG. 10. The graph of the breather lattice solution of type II. $N = 51$, $y = 25$, $m = 0.53$, and $\chi = 0.9$. The boundary conditions are the zero fixed ones $t = 0.0$.

Expression (50) gives the connection between the number of real spatial half-periods of the function $\mathrm{cn}\,(an, \chi)$ and the number of nodes N for the zero fixed boundary conditions.

Suppose that the length of the chain contains the integer number M of the spatial periods of the solution. Then $aN = 2\mathbb{K}(\chi)M$. Here the well known periodicity property of the Jacobi elliptic function (A11) is used. Periodic boundary conditions are satisfied if

$$a = \frac{2\mathbb{K}\,(\chi)}{N}M, \quad M = 1, 2, 3, \ldots, \tag{51}$$

$$N + M = 2l, \quad l = 1, 2, 3, \ldots. \tag{52}$$

If N is even (odd), then M is even (odd) (Figs. 11–13).

Thus, the solution (41) has five parameters a, q, A, m, χ, and also depends on such integer parameters as the number of lattice sites N and the number of spatial half-periods y (or the number of spatial periods M). Parameters q, A, m are defined by the expressions (42) and (43). In addition, the conditions (44) and (45) are imposed on the parameters a, χ, m. Using the boundary conditions the expression for the parameter a is obtained. In the case of the zero fixed boundary conditions the

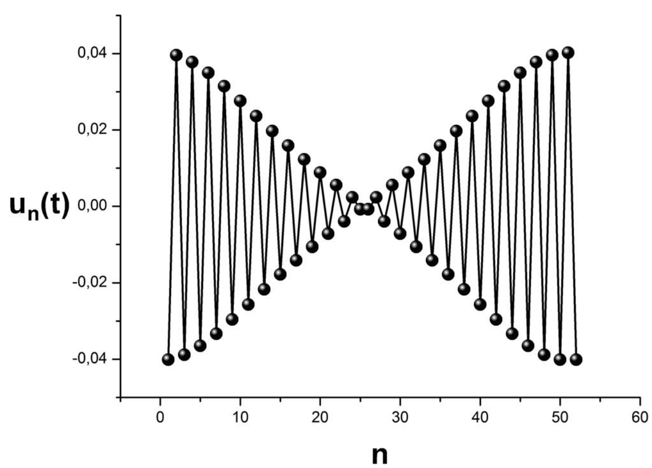

FIG. 11. The graph of the breather lattice solution of type II. $N = 51$, $M = 1$, $m = 0.0008$, and $\chi = 0.9$. The boundary conditions are periodic. $t = 0.0$.

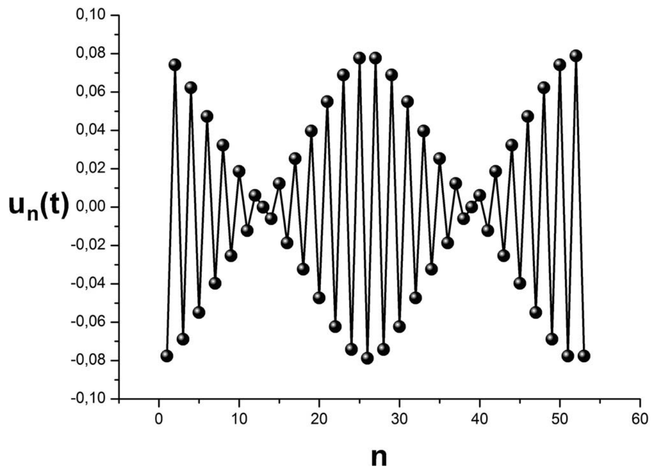

FIG. 12. The graph of the breather lattice solution of type II. $N = 52$. The boundary conditions are periodic. $t = 0.0$. $M = 2$, $m = 0.003$, and $\chi = 0.9$.

expression (49) holds, and in the case of the periodic boundary conditions parameter a is a function of the number of sites N and the number of the spatial periods M (51). For zero fixed boundary conditions the number of lattice sites N must be odd, the number of spatial half-periods y can be either odd or even. For periodic boundary conditions if N is even then the number of spatial periods M is even, if N is odd then M is odd.

Taking into account the boundary conditions the solution (41) is defined by one independent parameter χ and by such integer parameters as the number of lattice sites N and the number of spatial half-periods y (or the number of spatial periods M).

Below the dependences of the cyclic frequency of oscillations described by Eqs. (41)–(43) on energy for the periodic and zero fixed boundary conditions have been obtained. The cyclic frequency of the nonlinear oscillations (41)–(43) is defined by (46) and (42). Because the energy is the integral of motion it can be calculated for any arbitrary moment of time. It is convenient to choose the moment of time for which the following condition is valid:

$$qt = (2l + 1)\,\mathbb{K}(m), \quad l = 0, \pm 1, \pm 2, \ldots . \tag{53}$$

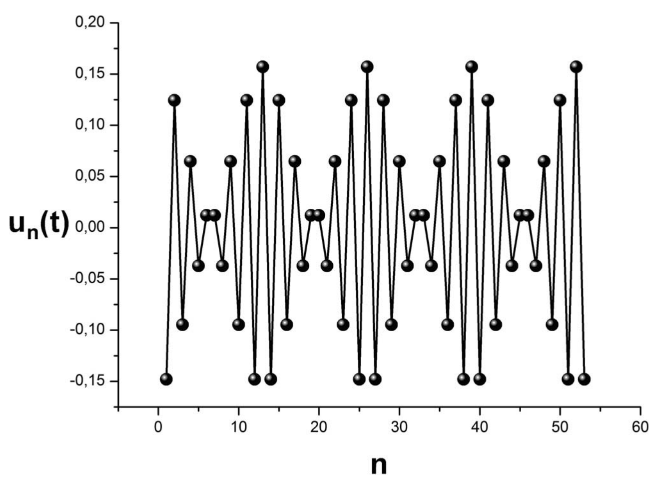

FIG. 13. The graph of the breather lattice solution of type II. $N = 52$. The boundary conditions are periodic. $t = 0.0$. $M = 4$, $m = 0.012$, and $\chi = 0.9$.

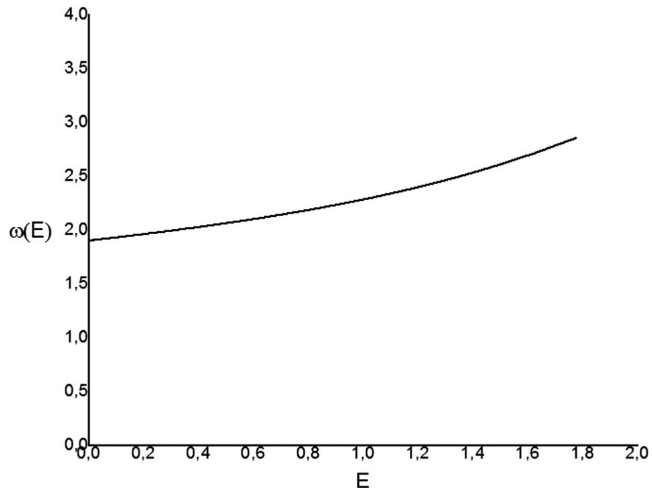

FIG. 14. The dependence of frequency on energy $\omega = \omega(E)$ for the breather lattice solution of type II. $N = 5$ and $M = 1$. The boundary conditions are periodic.

Then the energy of oscillations (41) equals

$$E = \frac{1}{2}\sum_{n=1}^{N} \ln\left[1 + A^2 m'^2 q^2 \mathrm{cn}^2\,(an,\,\chi)\right]. \tag{54}$$

For zero fixed boundary conditions the energy (11) for the N lattice sites can be calculated analytically. It equals

$$E = \frac{N-1}{4}\ln\left[1 + A^2 m'^2 q^2\right]. \tag{55}$$

For periodic boundary conditions the energy (54) can be calculated numerically for any number of sites N.

The dependences of the cyclic frequency of the nonlinear oscillations (41) on energy for the periodic and zero fixed boundary conditions are presented in Figs. 14 and 15, respectively.

The space-time evolution of the breather lattice solution of the type II for the ideal (Fig. 16), dissipative with $\lambda = 0.1$ (Fig. 17), and driven-damped with $\lambda = 0.1$, $F_i = 0.5\cos(\omega t)$, ($i = 5$,

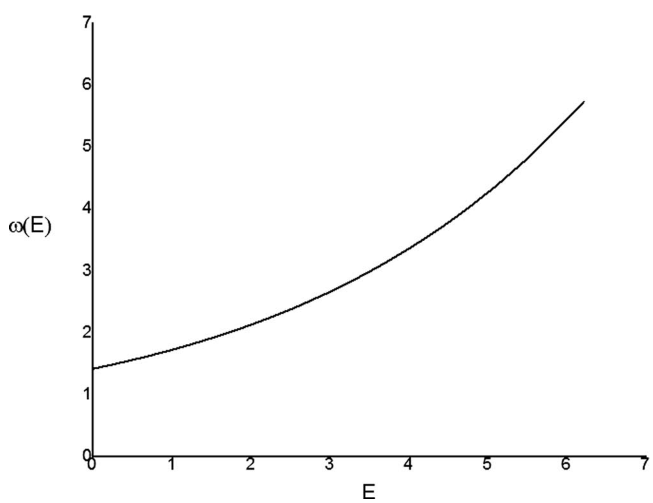

FIG. 15. The dependence of frequency on energy $\omega = \omega(E)$ for the breather lattice solution of type II. $N = 5$ and $M = 1$. The boundary conditions are the zero fixed ones.

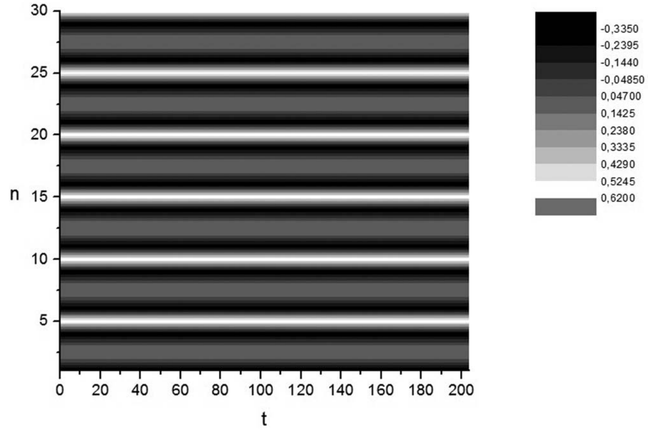

FIG. 16. A space time contour plot $u_n(t)$. Initial condition is the breather lattice solution of type II. $N = 30$, $M = 6$, $\chi = 0.99$, and $t_0 = 0$. Time is measured in the periods of the oscillation (41). Boundary conditions are periodic. The dynamics of the ideal lattice. $\lambda = 0$ and $F = 0$.

10, 15, 20, 25, 30) (Fig. 18) lattice has been investigated. The external forces are applied to the center-of-mass of each breather in the lattice. The frequency of the external force equals to the frequency of the breather lattice solution of type II. The periodic boundary conditions have been used. Simulations demonstrate the stability of the solution in the ideal lattice and instability in the dissipative lattice. The lifetime of breather lattice solution of the type II in the dissipative lattice can be extended through the concurrent application of ac driving terms.

V. SOLITON LATTICE SOLUTION

For Eq. (5) one more type of the periodic solution can be obtained

$$u_n(t) = \arctan\left[A\frac{\mathrm{sn}\,(an+bt,\,\chi)}{\mathrm{cn}\,(an+bt,\,\chi)}\mathrm{dn}\,(pn+qt,m)\right], \tag{56}$$

where

$$b = \pm\frac{2s_2c_2d_1}{s_2^2d_1^2 - A^2c_2^2}, \quad q = \pm\frac{\chi^2}{m^2}\frac{2s_1c_1d_2}{s_2^2d_1^2 - A^2c_2^2}, \tag{57}$$

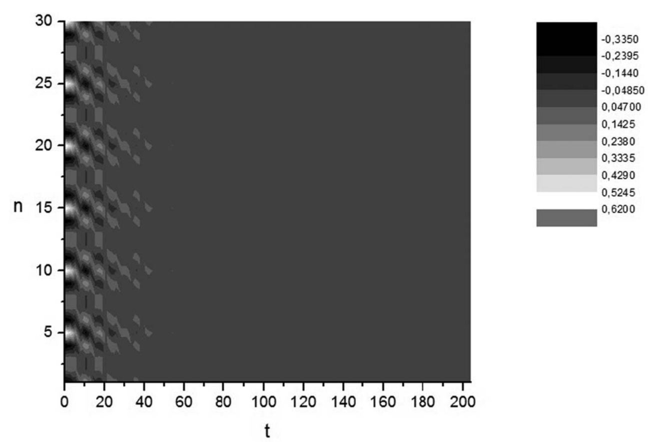

FIG. 17. A space time contour plot $u_n(t)$. Initial condition is the breather lattice solution of type II. $N = 30$, $M = 6$, $\chi = 0.99$, and $t_0 = 0$. Time is measured in the periods of the oscillation (41). Boundary conditions are periodic. The dynamics of the lattice with dissipation. $\lambda = 0.1$ and $F = 0$.

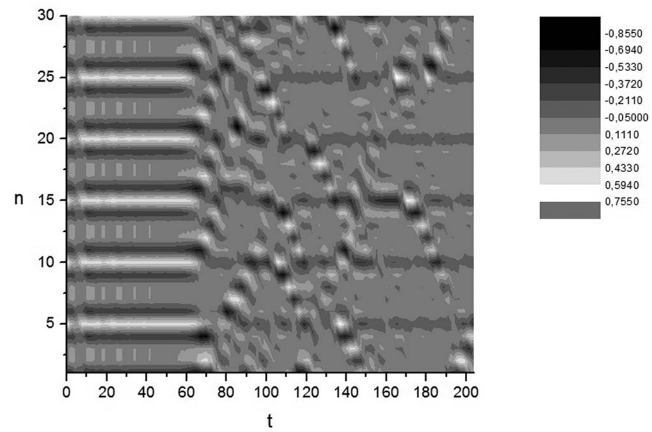

FIG. 18. A space time contour plot $u_n(t)$. Initial condition is the breather lattice solution of type II. $N = 30$, $M = 6$, $\chi = 0.99$, and $t_0 = 0$. Time is measured in the periods of the oscillation (41). Boundary conditions are periodic. The dynamics of the lattice with dissipation and point external periodic forces acting on the center-of-mass of the each breather. $\lambda = 0.1$, $F_i = 0.5\cos(\omega t)$, (i = 5, 10, 15, 20, 25, 30). The frequency of the external force equals to the frequency of the breather lattice solution of type II.

$$A = \sqrt{\frac{\chi'}{m'}} = \frac{d_1}{d_2}. \tag{58}$$

For $\chi \to 1$, $m \to 1$ the solution (56) reduces to the two-parametric soliton solution:

$$u_n(t) = \arctan\left[\frac{\cosh(p/2)}{\cosh(a/2)}\frac{\sinh(an+bt)}{\cosh(pn+qt)}\right], \tag{59}$$

where

$$\begin{aligned} b &= \pm 2\sinh(p/2)\cosh(a/2), \\ q &= \pm 2\sinh(a/2)\cosh(p/2). \end{aligned} \tag{60}$$

If we set $b = 0$, $\forall a$ for the solution (56)–(58), then $\mathrm{sn}(p/2, m)\,\mathrm{cn}(p/2, m) = 0$. If $\mathrm{sn}(p/2, m) = 0$, $\mathrm{cn}(p/2, m) \neq 0$ then

$$p = 4\mathbb{K}(m)\,l, \quad l = 0, \pm 1, \pm 2, \ldots. \tag{61}$$

Using the periodicity property of the Jacobi elliptic function (A12) the limiting case of the form of the nonlinear oscillations, namely, the soliton-soliton lattice solution, is obtained.

$$u_n(t) = \arctan\left[A\frac{\mathrm{sn}(an,\chi)}{\mathrm{cn}(an,\chi)}\mathrm{dn}(qt, m)\right], \tag{62}$$

$$q = \pm 2\frac{\chi^2}{m^2}\frac{s_1 c_1}{d_1^2}, \tag{63}$$

$$A = \sqrt{\frac{\chi'}{m'}} = d_1. \tag{64}$$

Obviously the period of oscillations for the expression (62) equals $T = 2\mathbb{K}(m)/q$ and the cyclic frequency equals

$$\omega = \frac{2\pi}{T} = \frac{\pi}{\mathbb{K}(m)}q. \tag{65}$$

For $\chi \to 1$, $m \to 1$ (62) reduces to the soliton-soliton solution in the system of reference connected with the center-of-mass of the solitons:

$$u_n^{(ss)} = \arctan\left[\frac{1}{\cosh(a/2)}\frac{\sinh an}{\cosh qt}\right], \tag{66}$$

$$q = \pm 2 \sinh (a/2) . \tag{67}$$

Because the function $\arctan(\theta)$ in (62) has the infinite discontinuity for $\theta_l = \frac{\pi}{2}(2l+1)$, $l = 0, \pm 1, \pm 2, \ldots$ it will be convenient to investigate the time derivative of the solution (62). For the model of the electrical transmission line the time derivative of the solution (62) means the current strength on the nth lattice site I_n:

$$\dot{u}_n(t) = \frac{-Aqm^2 \frac{\mathrm{sn}(an,\chi)}{\mathrm{cn}(an,\chi)} \mathrm{cn}(qt,m)\,\mathrm{sn}(qt,m)}{\left(1 + A^2 \frac{\mathrm{sn}^2(an,\chi)}{\mathrm{cn}^2(an,\chi)} \mathrm{dn}^2(qt,m)\right)}. \tag{68}$$

Zero fixed boundary conditions are satisfied if $\mathrm{sn}(a,\chi)/\mathrm{cn}(a,\chi) = 0$ and $\mathrm{sn}(aN,\chi)/\mathrm{cn}(aN,\chi) = 0$. Then

$$a = 2\mathbb{K}(\chi)(2l_1+1), \quad l_1 = 0, 1, 2, \ldots, \tag{69}$$

$$aN = 2\mathbb{K}(\chi)(2l_2+1), \quad l_2 = 1, 2, 3, \ldots. \tag{70}$$

For simplicity we set

$$a = 2\mathbb{K}(\chi). \tag{71}$$

Suppose that the length of the chain contains the integer number M of the real spatial periods of the solution. Then $aN = 2\mathbb{K}(\chi)M$ ((A10) and (A11)). Periodic boundary conditions are satisfied if

$$a = \frac{2\mathbb{K}(\chi)}{N} M, \quad M = 1, 2, 3, \ldots. \tag{72}$$

The space-time evolution of the soliton-soliton lattice solution for the ideal (Fig. 19) and dissipative (Fig. 20) lattice with $\lambda = 0.01$ has been investigated. The periodic boundary conditions have been used. Simulations demonstrate the stability of the solution in the ideal lattice and instability in the dissipative lattice.

If we set $q = 0$, $\forall p$ for the solution (56)–(58), then $\mathrm{sn}(a/2,\chi)\,\mathrm{cn}(a/2,\chi) = 0$. If $\mathrm{sn}(a/2,\chi) = 0$, $\mathrm{cn}(a/2,\chi) \neq 0$ then

$$a = 4\mathbb{K}(\chi)\,l, \quad l = 0, \pm 1, \pm 2, \ldots. \tag{73}$$

Using the periodicity properties of the Jacobi elliptic functions ((A10) and (A11)) the limiting case of the form of the soliton-antisoliton lattice solution was obtained:

$$u_n(t) = \arctan\left[A \frac{\mathrm{sn}(bt,\chi)}{\mathrm{cn}(bt,\chi)} \mathrm{dn}(pn,m)\right], \tag{74}$$

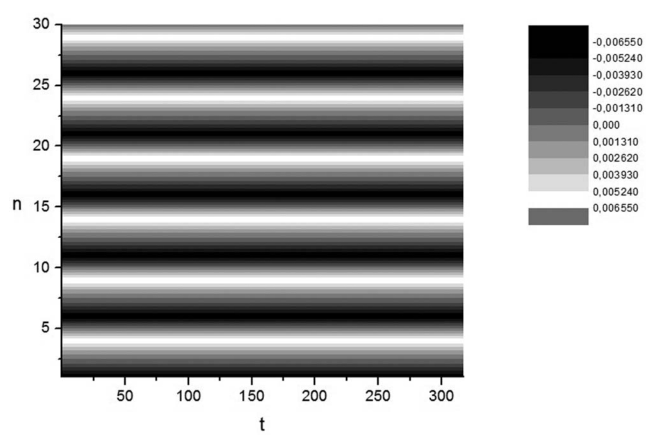

FIG. 19. A space time contour plot $\dot{u}_n(t)$. Initial condition is the soliton-soliton lattice solution. $N = 30$, $M = 6$, $\chi = 0.99$, and $t_0 = 0$. Time is measured in the periods of the oscillation (68). Boundary conditions are periodic. The dynamics of the ideal lattice. $\lambda = 0$ and $F = 0$.

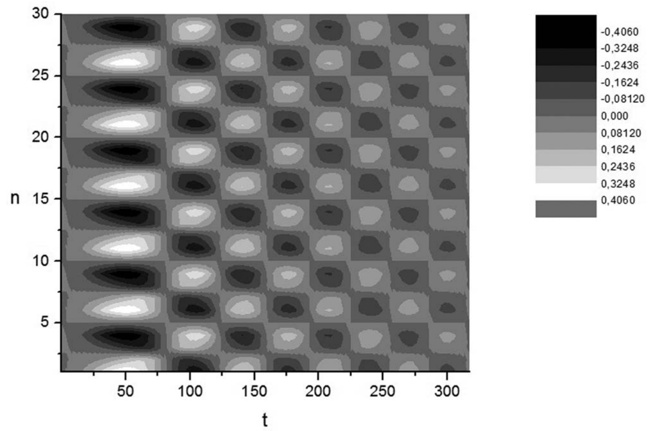

FIG. 20. A space time contour plot $\dot{u}_n(t)$. Initial condition is the soliton-soliton lattice solution. $N = 30$, $M = 6$, $\chi = 0.99$, and $t_0 = 0$. Time is measured in the periods of the oscillation (68). Boundary conditions are periodic. The dynamics of the lattice with dissipation. $\lambda = 0.01$ and $F = 0$.

$$b = \pm\frac{2s_2c_2d_2^2}{s_2^2d_2^2 - c_2^2}, \tag{75}$$

$$A = \sqrt{\frac{\chi'}{m'}} = \frac{1}{d_2}. \tag{76}$$

Obviously the period of oscillations for the expression (74) equals $T = 2\mathbb{K}(\chi)/b$ and the cyclic frequency

$$\omega = \frac{2\pi}{T} = \frac{\pi}{\mathbb{K}(\chi)}b. \tag{77}$$

For $\chi \to 1$, $m \to 1$ the expression (74) reduces to the soliton-antisoliton solution in the system of reference connected with the center-of-mass of the solitons:

$$u_n^{(s\bar{s})} = \arctan\left[\cosh(p/2)\,\frac{\sinh bt}{\cosh pn}\right], \tag{78}$$

$$b = \pm 2\sinh(p/2). \tag{79}$$

The time derivative of the solution (74) equals

$$\dot{u}_n(t) = \frac{Ab\mathrm{dn}(bt,\chi)\,\mathrm{dn}(pn,m) + Ab\frac{\mathrm{sn}^2(bt,\chi)\mathrm{dn}(bt,\chi)}{\mathrm{cn}^2(bt,\chi)}\mathrm{dn}(pn,m)}{1 + A^2\frac{\mathrm{sn}^2(bt,\chi)}{\mathrm{cn}^2(bt,\chi)}\mathrm{dn}^2(pn,m)}. \tag{80}$$

Because the function $\mathrm{dn}(\theta, m) \neq 0,\ \forall\theta$ the solution (74), (80) does not satisfy zero fixed boundary conditions. Suppose that the length of the chain contains the integer number M of the spatial periods of the solution. Then $pN = 2\mathbb{K}(m)M$. Periodic boundary conditions are satisfied if

$$p = \frac{2\mathbb{K}(m)}{N}M, \quad M = 1, 2, 3, \ldots. \tag{81}$$

The space-time evolution of the soliton-antisoliton lattice solution for the ideal (Fig. 21) and dissipative (Fig. 22) lattice with $\lambda = 0.1$ has been investigated. The periodic boundary conditions have been used. Simulations demonstrate the stability of the solution in the ideal lattice and instability in the dissipative lattice.

After the changing of the variables in Eq. (56)

$$a = \tilde{a} + 2i\mathbb{K}(\chi), \quad p = \tilde{p} + 2i\mathbb{K}(m), \tag{82}$$

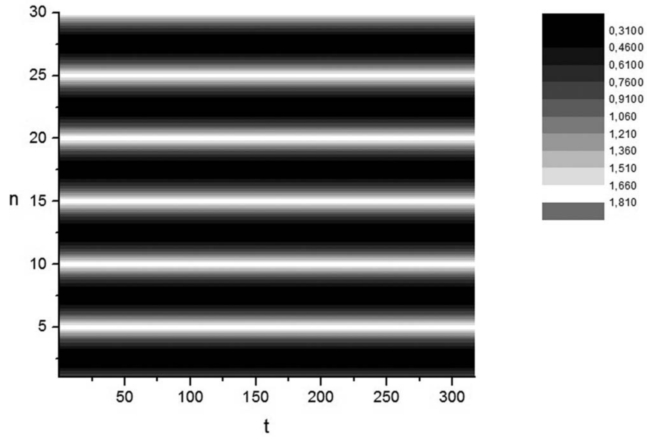

FIG. 21. A space time contour plot $\dot{u}_n(t)$. Initial condition is the soliton-antisoliton lattice solution. $N = 30$, $M = 6$, and $m = 0.99$. Time is measured in the periods of the oscillation (80). Boundary conditions are periodic. The dynamics of the ideal lattice $\lambda = 0$ and $F = 0$.

the new solution of Eq. (5) is found

$$u_n(t) = \arctan\left[\tilde{A}\frac{\mathrm{sn}\left(\tilde{a}n + \tilde{b}t, \chi\right)}{\mathrm{cn}\left(\tilde{a}n + \tilde{b}t, \chi\right)}\mathrm{dn}\left(\tilde{p}n + \tilde{q}t, m\right)\right], \tag{83}$$

$$\tilde{b} = \pm\frac{2\tilde{s}_1\tilde{c}_1\tilde{d}_2}{\tilde{c}_1^2 - \tilde{A}^2\tilde{s}_1^2\tilde{d}_2^2}, \quad \tilde{q} = \pm\frac{2\tilde{s}_2\tilde{c}_2\tilde{d}_1}{\tilde{c}_1^2 - \tilde{A}^2\tilde{s}_1^2\tilde{d}_2^2}, \tag{84}$$

$$\tilde{A} = \sqrt{\frac{\chi'}{m'}} = \frac{\tilde{s}_2\tilde{c}_1}{\tilde{s}_1\tilde{c}_2}, \tag{85}$$

where the following notations are used:

$$\begin{aligned} \tilde{s}_1 &\equiv \mathrm{sn}\left(\tilde{a}/2, \chi\right), \tilde{c}_1 \equiv \mathrm{cn}\left(\tilde{a}/2, \chi\right), \tilde{d}_1 \equiv \mathrm{dn}\left(\tilde{a}/2, \chi\right), \\ \tilde{s}_2 &\equiv \mathrm{sn}\left(\tilde{p}/2, m\right), \tilde{c}_2 \equiv \mathrm{cn}\left(\tilde{p}/2, m\right), \tilde{d}_2 \equiv \mathrm{dn}\left(\tilde{p}/2, m\right). \end{aligned} \tag{86}$$

Here the periodicity properties of the Jacobi elliptic functions (A10)–(A12) have been used. For $\chi \to 1$, $m \to 1$ the solution (83) reduces to the two-parametric soliton solution.

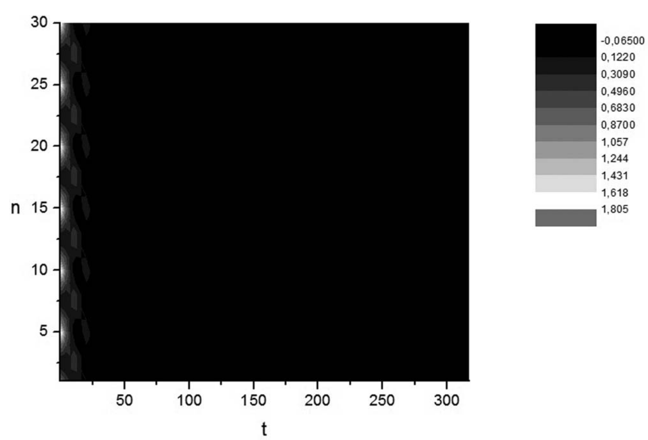

FIG. 22. A space time contour plot $\dot{u}_n(t)$. Initial condition is the soliton-antisoliton lattice solution. $N = 30$, $M = 6$, and $m = 0.99$. Time is measured in the periods of the oscillation (80). Boundary conditions are periodic. The dynamics of the lattice with dissipation $\lambda = 0.1$ and $F = 0$.

VI. CONCLUSIONS

The new classes of periodic solutions of the nonlinear self-dual network equations and equivalent Hirota lattice equation describing the nonlinear spatial periodic oscillations and waves, expressed in terms of the Jacobi elliptic functions have been found. The obtained solutions could be identified as the breather lattice and soliton lattice solutions, since in the appropriate limit they reduce to single breather, soliton-soliton, and soliton-antisoliton solutions. In the linear limit these solutions reduce to the linear running and standing waves. The new solutions have been found for the infinite lattice and for the finite-size lattice with the periodic and fixed boundary conditions. The dependences of the frequency on the energy of the breather lattice solutions of types I and II for the finite-size lattice have been obtained.

The dynamical stability of the solutions has been examined through direct numerical simulations by means of a 7th order Runge-Kutta scheme. Numerical simulations of the soliton lattice demonstrate its stability and the breather lattice metastability in the ideal lattice, and their instabilities in the dissipative lattice. It has been shown that the lifetime of the breather lattice solution of type II in the dissipative lattice can be extended through the application of ac driving terms.

APPENDIX A: THE PROPERTIES OF THE JACOBI ELLIPTIC FUNCTIONS USED IN THE PAPER

The derivatives of the Jacobi elliptic functions on the argument:

$$\frac{d}{d\theta}\mathrm{sn}\,(\theta,\chi) = \mathrm{cn}\,(\theta,\chi)\,\mathrm{dn}\,(\theta,\chi)\,, \tag{A1}$$

$$\frac{d}{d\theta}\mathrm{cn}\,(\theta,\chi) = -\mathrm{sn}\,(\theta,\chi)\,\mathrm{dn}\,(\theta,\chi)\,, \tag{A2}$$

$$\frac{d}{d\theta}\mathrm{dn}\,(\theta,\chi) = -\chi^2\mathrm{sn}\,(\theta,\chi)\,\mathrm{cn}\,(\theta,\chi)\,. \tag{A3}$$

The Jacobi elliptic functions for the sum and difference of the arguments

$$\mathrm{sn}\,(\alpha\pm\beta,\chi) = \frac{\mathrm{sn}\,(\alpha,\chi)\,\mathrm{cn}\,(\beta,\chi)\,\mathrm{dn}\,(\beta,\chi)\pm\mathrm{sn}\,(\beta,\chi)\,\mathrm{cn}\,(\alpha,\chi)\,\mathrm{dn}\,(\alpha,\chi)}{1-\chi^2\mathrm{sn}^2\,(\alpha,\chi)\,\mathrm{sn}^2\,(\beta,\chi)}, \tag{A4}$$

$$\mathrm{cn}\,(\alpha\pm\beta,\chi) = \frac{\mathrm{cn}\,(\alpha,\chi)\,\mathrm{cn}\,(\beta,\chi)\mp\mathrm{sn}\,(\alpha,\chi)\,\mathrm{sn}\,(\beta,\chi)\,\mathrm{dn}\,(\alpha,\chi)\,\mathrm{dn}\,(\beta,\chi)}{1-\chi^2\mathrm{sn}^2\,(\alpha,\chi)\,\mathrm{sn}^2\,(\beta,\chi)}, \tag{A5}$$

$$\mathrm{dn}\,(\alpha\pm\beta,\chi) = \frac{\mathrm{dn}\,(\alpha,\chi)\,\mathrm{dn}\,(\beta,\chi)\mp\chi^2\mathrm{sn}\,(\alpha,\chi)\,\mathrm{sn}\,(\beta,\chi)\,\mathrm{cn}\,(\alpha,\chi)\,\mathrm{cn}\,(\beta,\chi)}{1-\chi^2\mathrm{sn}^2\,(\alpha,\chi)\,\mathrm{sn}^2\,(\beta,\chi)}. \tag{A6}$$

The Jacobi elliptic functions reduce to the trigonometric and hyperbolic functions if $\chi\to 0$ and if $\chi\to 1$:

$$\mathrm{sn}\,(\theta,\chi) = \begin{cases} \sin\,(\theta)\,,\ \chi\to 0 \\ \tanh\,(\theta)\,,\ \chi\to 1 \end{cases}, \tag{A7}$$

$$\mathrm{cn}\,(\theta,\chi) = \begin{cases} \cos\,(\theta)\,,\ \chi\to 0 \\ \mathrm{sech}\,(\theta)\,,\ \chi\to 1 \end{cases}, \tag{A8}$$

$$\mathrm{dn}\,(\theta,\chi) = \begin{cases} 1,\ \chi\to 0 \\ \mathrm{sech}\,(\theta)\,,\ \chi\to 1 \end{cases}. \tag{A9}$$

The elliptic Jacobi functions $\mathrm{sn}\,(\theta,\chi)$, $\mathrm{cn}\,(\theta,\chi)$, $\mathrm{dn}\,(\theta,\chi)$ have the properties of the periodicity. They have the real and pure imaginary periods. The following relations are valid:

$$\mathrm{sn}\left(\theta+2\alpha\mathbb{K}+2\beta i\mathbb{K}'\right) = (-1)^{\alpha}\,\mathrm{sn}\,(\theta)\,, \tag{A10}$$

$$\mathrm{cn}\left(\theta + 2\alpha\mathbb{K} + 2\beta i\mathbb{K}'\right) = (-1)^{\alpha+\beta}\,\mathrm{cn}\,(\theta)\,, \tag{A11}$$

$$\mathrm{dn}\left(\theta + 2\alpha\mathbb{K} + 2\beta i\mathbb{K}'\right) = (-1)^{\beta}\,\mathrm{dn}\,(\theta)\,, \tag{A12}$$

where $\mathbb{K}$ and $\mathbb{K}'$ are the complete elliptic integrals of the first kind.

$$\mathbb{K} = \mathbb{K}\,(\chi) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\chi^2\sin^2\varphi}}, \tag{A13}$$

$$\mathbb{K}' = \mathbb{K}\left(\chi'\right) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-(\chi')^2\sin^2\varphi}}, \tag{A14}$$

$\alpha, \beta = 0, \pm 1, \pm 2, \ldots$ are the integer numbers, i is the imaginary unit.

APPENDIX B: THE DERIVATION PROCEDURE OF THE NONLINEAR PERIODIC WAVES SOLUTIONS

For the derivation of the nonlinear periodic waves solutions (14), (35), and (56) of the nonlinear self-dual network equations (SDNE) (1) or the Hirota lattice equation (5) we will use the more simple Eq. (3) which is equivalent to (1) and (5):

$$\frac{du_n}{dt} = \tan\left(u_{n-1/2} - u_{n+1/2}\right). \tag{B1}$$

For brevity we will use the following notation:

$$\begin{aligned} S_1 &\equiv \mathrm{sn}\,(\Phi, \chi)\,, C_1 \equiv \mathrm{cn}\,(\Phi, \chi)\,, D_1 \equiv \mathrm{dn}\,(\Phi, \chi)\,, \\ S_2 &\equiv \mathrm{sn}\,(\Theta, m)\,, C_2 \equiv \mathrm{cn}\,(\Theta, m)\,, D_2 \equiv \mathrm{dn}\,(\Theta, m)\,, \end{aligned} \tag{B2}$$

where

$$\Phi = an + bt, \quad \Theta = pn + qt. \tag{B3}$$

It is convenient to make the following change of the variables in Eq. (B1):

$$u_n = \arctan\,(v_n)\,. \tag{B4}$$

Then Eq. (B1) takes the form

$$\frac{1}{1+v_n^2}\frac{dv_n}{dt} = \frac{v_{n-1/2} - v_{n+1/2}}{1 + v_{n-1/2}v_{n+1/2}}. \tag{B5}$$

To obtain the breather lattice solution of type I (14) we start with the ansatz

$$v_n = A\mathrm{sn}\,(\Phi, \chi)\,\mathrm{dn}\,(\Theta, m)\,, \tag{B6}$$

with $A = const$, where $\mathrm{sn}\,(\Phi, \chi)$ and $\mathrm{dn}\,(\Theta, m)$ are Jacobi elliptic functions with modulus χ and m, respectively, and Φ and Θ are defined by (B3). Using the notation (B2) expression (B6) takes the form

$$v_n = AS_1D_2. \tag{B7}$$

This ansatz is inspired by the structural relationship between a single breather solution of the sine-Gordon equation and that of the SDNE (19) as well as by the functional similarity of the sine-Gordon breather lattice solution.[17,18] Substituting the ansatz (B7) in Eq. (B5) and equating the expressions corresponding to the same combinations of the elliptic functions (B2) we obtain the system of the algebraic equations for the parameters of the solution:

$$C_1D_1D_2: \quad b\left(1 - A^2y^2\delta^2\right) = -2y\delta, \tag{B8}$$

$$S_2^2 C_1 D_1 D_2: \quad -bA^2 y^2 T = 2m^2\varepsilon^2 y\delta + 2bm^2\varepsilon^2, \tag{B9}$$

$$S_2^4 C_1 D_1 D_2: \quad A^2 y^2 \beta^2 = \varepsilon^4, \tag{B10}$$

$$\begin{aligned} &S_1^2 C_1 D_1 D_2: \\ &bA^2\delta^2 V = 2\chi^2 y^3\delta - 2A^2 y\delta + 2b\chi^2 y^2, \end{aligned} \tag{B11}$$

$$\begin{aligned} &S_1^2 S_2^2 C_1 D_1 D_2: \quad bA^2 TV = -2\chi^2 m^2 y^3\varepsilon^2\delta + \\ &+2A^2 m^2 y\delta + 2A^2 m^2\varepsilon^2 y\delta - 4b\chi^2 m^2 y^2\varepsilon^2, \end{aligned} \tag{B12}$$

$$\begin{aligned} &S_1^2 S_2^4 C_1 D_1 D_2: \\ &bA^2 V\beta^2 = -2A^2\varepsilon^2 y\delta + 2b\chi^2 y^2\varepsilon^4, \end{aligned} \tag{B13}$$

$$S_1^4 C_1 D_1 D_2: \quad -bA^2\delta^2 = 2A^2 y\delta - b\chi^2 y^2, \tag{B14}$$

$$\begin{aligned} &S_1^4 S_2^2 C_1 D_1 D_2: \quad -bA^2 T = -2A^2 m^2 y\varepsilon^2\delta - \\ &-2A^2 m^2 y\delta + 2b\chi^2 y^2 m^2\varepsilon^2, \end{aligned} \tag{B15}$$

$$S_1^4 S_2^4 C_1 D_1 D_2: -bA^2\beta^2 = 2A^2 y\varepsilon^2\delta - b\chi^2 y^2\varepsilon^4, \tag{B16}$$

$$S_1 S_2 C_2: \quad -q\left(1 - A^2 y^2\delta^2\right) = 2X\beta, \tag{B17}$$

$$S_1 S_2^3 C_2: \quad qA^2 y^2 T = -2m^2 X\beta\varepsilon^2 - 2qm^2\varepsilon^2, \tag{B18}$$

$$S_1 S_2^5 C_2: \quad A^2 y^2\beta^2 = \varepsilon^4, \tag{B19}$$

$$\begin{aligned} &S_1^3 S_2 C_2: \\ &-qA^2\delta^2 V = -2\chi^2 y^2 X\beta + 2A^2 X\beta - 2q\chi^2 y^2, \end{aligned} \tag{B20}$$

$$\begin{aligned} &S_1^3 S_2^3 C_2: \quad -qA^2 TV = 2\chi^2 m^2 y^2\varepsilon^2 X\beta - \\ &-2A^2 m^2\varepsilon^2 X\beta - 2A^2 m^2 X\beta + 4q\chi^2 m^2 y^2\varepsilon^2, \end{aligned} \tag{B21}$$

$$S_1^3 S_2^5 C_2: -qA^2 V\beta^2 = 2A^2 X\beta\varepsilon^2 - 2q\chi^2 y^2\varepsilon^4, \tag{B22}$$

$$S_1^5 S_2 C_2: \quad qA^2\delta^2 = -2A^2 X\beta + q\chi^2 y^2, \tag{B23}$$

$$\begin{aligned} &S_1^5 S_2^3 C_2: qA^2 T = 2A^2 m^2 X\beta + \\ &+2A^2 m^2\varepsilon^2 X\beta - 2qm^2\chi^2 y^2\varepsilon^2, \end{aligned} \tag{B24}$$

$$S_1^5 S_2^5 C_2: \quad qA^2 = -2A^2\varepsilon^2 X\beta + q\chi^2 y^2\varepsilon^4, \tag{B25}$$

where the following notation is used:

$$X = c_1 d_1, \quad y = s_1, \quad \delta = d_2, \quad \beta = s_2 c_2, \quad \varepsilon = s_2, \tag{B26}$$

$$T = -m^2 + m^4 s_2^4, \quad V = 1 + \chi^2 s_1^4. \tag{B27}$$

Solving the system of Eqs. (B8)–(B25) we conclude that Eqs. (B8), (B9), (B11)–(B16) and (B17), (B18), (B20)–(B25) give the expressions (15), and Eqs. (B10) and (B19) give the expressions (16).

The solutions (35) and (56) have been derived in the similar way. Substituting the ansatzes

$$v_n = AC_1C_2. \tag{B28}$$

$$v_n = A\frac{S_1}{C_1}D_2 \tag{B29}$$

in Eq. (B5) and after analogous tedious algebraic manipulations we have found that Eqs. (35)–(37) and (56)–(58) are indeed the exact periodic solutions of Eqs. (B1) or (1) and (5).

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