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LETTER TO THE EDITOR

Bifurcation picture and stability of gap and out-gap discrete solitons

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The dynamics of a quaternary fragment of a discrete system of coupled nonlinear oscillators with modulated frequency parameters is investigated, and the stability of its gap and out-gap soliton-like excitations is studied. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2737564]

I. INTRODUCTION

The concept of the gap (Bragg) soliton first appeared in nonlinear optics.[1,2] These solitons can exist in nonlinear systems with spatial periodicity of some material parameters and possess the frequencies in the gap of the spectrum of linear excitations. The interest in this problem is due to the fact that the group velocity of linear waves, linear pulses, and solitons tends to zero at the boundary of the gap. The velocity of the optical pulse is reduced significantly in material domains with modulated parameters which are inserted into the optical fiber, and this effect could be used in nonlinear optical devices.

Solitons with frequencies outside of the gap, the out-gap solitons, were first studied in Ref. 3. Embedded solitons of this type have frequencies inside the band of linear excitations and nonzero asymptotics at infinity. Therefore, in contrast to the gap soliton, the out-gap soliton has infinite norm. The gap and out-gap solitons in discrete systems with alternating atomic or spin characteristics, e.g., in diatomic lattices and two-sublattice magnets, have been investigated by many authors. Many of the results obtained were similar to those for extended systems, while some finite-size models were solved exactly in special cases. However two important questions remain still open: (i) how does the gap soliton transform into the out-gap one at the boundary of the frequency gap of linear waves ("linear gap"), and (ii) what are the stability properties of the gap and out-gap solitons?

It is well known[4] that many aspects of soliton dynamics of nonlinear systems can be elucidated in the framework of models with a finite number of degrees of freedom. The simplest discrete modulated system that admits the existence of analogs of the gap and out-gap excitations is a ring of four coupled nonlinear oscillators with alternating frequency parameters.[5] At present this model is of great interest for low-temperature physics due to the topical problem of magnetic molecular nanoclusters.[6] It is known that in systems of finite size or with a finite number of degrees of freedom, quasi-solitons appear in a bifurcation manner beginning from the moment when the energy or system parameters, in particular, the frequency modulation depth γ,[5] exceed some threshold values. The scenario of the birth of the analog of the gap soliton in the quaternary model contains two bifurcations: at $\gamma_c=1.707$, where some new excitations appear in the "nonlinear gap" (see below), and at $\gamma_*=1.750$, where analogs of the gap and out-gap modes appear. This bifurcation pattern is qualitatively depicted in the inset of Fig. 3 in Ref. 5. These two bifurcations were first discovered numerically by L. Kroon,[7] who informed the authors of Ref. 5 about his results before the publication.[5] The exact bifurcation picture represented by the numerical results of Ref. 7 is shown in Fig. 1. The main goal of this paper is to reveal details of the scenario of transformation of the gap soliton analog into an out-gap soliton one and to analyze the stability of these nonlinear excitations.

II. THE MODEL

Nonlinear dynamics of a ring consisting of four coupled anharmonic oscillators (or classical spins with numbers n

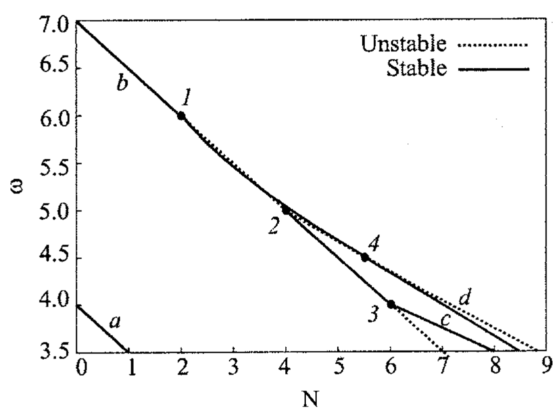

FIG. 1. Stationary solutions in the (ω,N) plane for $\gamma=1.75$. Solid (dotted) lines represent stable (unstable) regions of the solutions.

$=1,2,3,4$) with a periodic modulation of the frequency parameter is considered in the framework of discrete nonlinear Schrödinger equation (DNLSE):[5]

$$i\dot{\psi}_n - \omega_0^{(n)}\psi_n + \varepsilon(\psi_{n+1} + \psi_{n-1}) + |\psi_n|^2\psi_n = 0. \quad (1)$$

The corresponding Hamiltonian of the model has the form

$$H = \sum_n [\omega_0^{(n)}|\psi_n|^2 - \varepsilon(\psi_n\psi_{n-1}^* + \psi_n^*\psi_{n-1}) - |\psi_n|^4/2], \quad (2)$$

with canonically conjugate variables $\{i\psi_n\}$, $\{\psi_n^*\}$, and $\omega_0^{(n)} \equiv \omega_a(\omega_0^{(n)} \equiv \omega_b)$, when the index n is an odd (even) number. In addition to the Hamiltonian (2) also the norm (excitations number), defined by $N=\Sigma_n|\psi_n|^2$, is a conserved quantity for (1). The ratio $\gamma \equiv \omega_b/\omega_a$ (we suppose for definiteness $\gamma > 1$) reflects the depth of modulation. The transformation $\psi_n \to \sqrt{\varepsilon}\psi_n$, $\omega_0^{(n)} \to \varepsilon\omega_0^{(n)}$ is invoked in normalizing the coupling ε to unity. We will discuss the stationary states of the form $\psi_n^{(\omega)}(t)=\varphi_n \times \exp(-i\omega t)$ with real amplitudes φ_n. In the linear limit ($\psi_n \to 0$) the spectrum of normal modes contains only 4 frequencies for in-phase and antiphase oscillations with $\omega_{\min,\max}=[(\omega_a+\omega_b) \mp \sqrt{(\omega_b-\omega_a)^2+16}]/2$, and for the gap boundaries solutions with $\omega=\omega_a$ and with $\omega=\omega_b$. In the limit of a long chain these frequencies do not change, but the domains $(\omega_{\min},\omega_a)$ and $(\omega_b,\omega_{\max})$ transform into two bands of the spectrum with the gap (ω_a,ω_b). That is why in our simple model we will call this domain a linear gap. In the nonlinear case the frequencies of the aforementioned four "main nonlinear modes" decrease, and the "linear gap" transforms into a "nonlinear gap" with $\omega_{a,b}(N)=\omega_{a,b}-N_{a,b}/2$ (see the lines (*a*) and (*b*) in Figs. 1 and 2). The frequencies ω_a and ω_b at the boundaries of the gap correspond to the antiphase oscillations (↑0↓0) and (0⇑0⇓), respectively, where the zeros indicate immovable particles and the thickness of the arrows characterizes the relative amplitude of the oscillations. The frequencies for nonlinear oscillations depend not only on the parameter γ, but also on the amplitude and, hence, implicitly on the norm N and the energy E defined from the Hamiltonian (2). The "spectral" dependence $E=E(N)$ of the system is uniquely determined by the characteristic $\omega=\omega(N)$ due to the fulfillment of the relation $\omega=dE/dN$ for monochromatic oscillations. The most interesting for us are the upper boundary of the nonlinear gap (line *b* in Fig. 1) and the analog of the gap and out-gap modes (line *e* in Fig. 2), bifurcating from the boundary.

III. THE STABILITY PROBLEM

The stability of the solution $\psi_n^{(\omega)}(t)$ is analyzed by adding the perturbation $\varphi_n(t)=\alpha_n \exp(-i\Omega t)+\beta_n^* \exp(i\Omega^* t)$ to its time-independent amplitude φ_n. Linearization of Eq. (1) around the stationary solution yields the eigenvalue problem

$$(\omega - \omega_0^{(n)} + 2\varphi_n^2)\alpha_n + \alpha_{n+1} + \alpha_{n-1} + \varphi_n^2\beta_n = -\Omega\beta_n,$$

$$(\omega - \omega_0^{(n)} + 2\varphi_n^2)\beta_n + \beta_{n+1} + \beta_{n-1} + \varphi_n^2\alpha_n = \Omega\beta_n. \quad (3)$$

The linear stability of the stationary solution is equivalent to all eigenfrequencies Ω being real.

(i) Stability of the *b* mode. This problem can be solved analytically. Introducing the notation $A=\omega_a-\omega$, $B=\omega_b-\omega$, $\omega_0=(\omega_a+\omega_b)/2$ and the width $\Delta \equiv B-A=(\gamma-1)\omega_0$ of the linear gap, the solution for the upper boundary of the "nonlinear" gap (*b* mode) can be written[5] as $\{\varphi_n^{(b)}\}=\{0,\sqrt{B},0,-\sqrt{B}\}$ and the eigenvalues Ω are found to satisfy the equation

$$\Omega^2(\Omega^2 - A^2)(\Omega^4 - (A^2+8)\Omega^2 + 8AB + 16) = 0. \quad (4)$$

The root $\Omega=0$ corresponds to "phase mode" and the roots $\Omega=\pm A$ are real, whereas the remaining eigenvalues

$$\Omega = \pm\sqrt{(A^2+8)/2 \pm \sqrt{(A^2+8)^2/4 - 8(AB+2)}} \quad (5)$$

are complex whenever $A(A^3-16A-32\Delta)<0$. In the notation $\omega_\pm = \sqrt[3]{16\Delta \pm \sqrt{(16\Delta)^2-(16/3)^3}}$, this criterion leads to oscillatory instability[2] in the region $0<A<\omega_+ + \omega_-$, which appears through Krein collisions[8] and manifests itself through resonances of the internal modes. One may note from Fig. 1 that there exist two windows of instability for the *b* mode. The first one is bounded by the bifurcation point *1* of the gap *e* mode and the bifurcation point *2* in Fig. 1. For large γ this is the bifurcation point for the unstable *d* mode (Fig. 2). A second interval developing for $\Delta<4/(3\sqrt{3})$, is ruled out by the constraint $B=A+\Delta>0$ and is bounded by the point *3* for the bifurcation of the *c* mode $\{\varphi_n^{(c)}\}=\{\sqrt{A},\sqrt{B},-\sqrt{A},-\sqrt{B}\}$ (Ref. 5) at the frequency $\omega=\omega_a$ (the low boundary of the linear gap). If $AB+2<0$ is satisfied, two of the eigenvalues (5) are not real. Applying the dependence $\omega_b(N)=\gamma\omega_a-N/2$ of frequency on norm N, inequality gives an interval $|\omega_b(N)-\omega_0|<\sqrt{(\Delta/2)^2-2}$, for which the solutions $\{\varphi_n^{(b)}\}$ are unstable, if and only if $\Delta>2\sqrt{2}\equiv\Delta_c$. For large γ (see Fig. 2) the *c* mode is stable only in the vicinity of point 3. The lower boundary of the nonlinear gap corresponds to the *a* mode, which is linearly stable for all values of parameter γ.

(ii) The stability of the gap and out-gap modes. First of all we notice that our investigations of the gap solitons in the systems with 4, 6, 8, 10, 12, 16 particles have shown that the dependence of the gap and out-gap solution frequencies on the norm changes qualitatively in the same manner when the number of particles grows or the parameter γ grows. The following result is the most important: the dependence $\omega=\omega(N)$ does not embed into the lower zone of the linear-wave spectrum and remains inside the nonlinear gap of the spectrum, but it transforms essentially at the frequency of the lower boundary of the linear gap ($\omega=4$ in Fig. 2). At this frequency value the transformation of the gap soliton into the

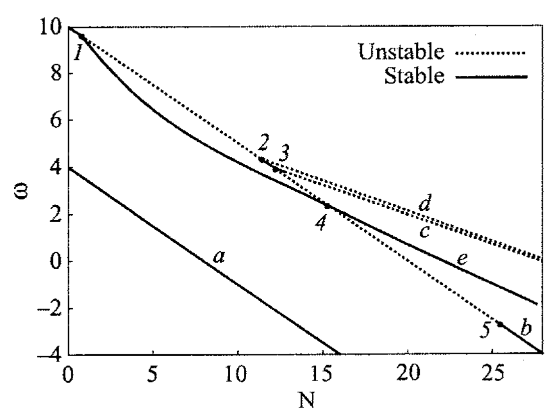

FIG. 2. Stationary solutions in the (ω,N) plane for $\gamma=2.5$, which is the threshold for the linear stability of the gap and out-gap solitons.

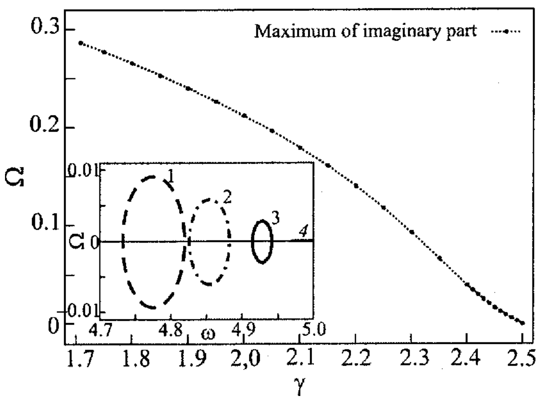

FIG. 3. The development of the Krein instability for the gap and out-gap breathers versus γ: 2.47 (1), 2.48 (2), 2.49 (3), $\geqslant 2.5$ (4). For $\gamma > 2.35$ the oscillatory instability enters the gap regions ($\omega < 4$), and the resonance ultimately vanishes at $\gamma = 2.5$ ($\omega = 5$). Close to the threshold of the stability the maximum value of the imaginary part of Ω shows a linear scaling in ω (and γ).

out-gap one takes place. There are not analytical expressions for the gap and out-gap solitons, except the case $\gamma = 2.5$.[9] We studied the stability of these excitations numerically in the framework of Eqs. (3) with the use of numerical solutions for φ_n. The results are shown in Fig. 3. There exists the window of the Krein (oscillatory) instability of the gap and out-gap solitons analogs with nonzero imaginary part of parameter Ω. But Im Ω tends to zero for $\gamma \to 2.5$ (large depth of modulation). This justifies the expectation that the gap and out-gap solitons are stable in the large modulated systems. In the inset of Fig. 3 the transformation of the window of instability is presented: it lies inside the linear gap, and the frequencies of the window and its width decrease with the growth of parameter γ.

IV. CONCLUSION

The analogs of gap and out-gap solitons have been studied in the quaternary fragment of discrete modulated nonlinear system of coupled oscillators. It has been demonstrated that transformation of such monochromatic soliton-like solutions and their stability depend essentially on the value of the modulating parameter γ. After two bifurcations at $\gamma = 1.707$ and $\gamma = 1.750$ a unified dependence of the soliton frequency ω on the norm N for gap and out-gap solitons is formed. The gap soliton transforms into an out-gap one at the lower boundary of the "linear gap" of the spectrum, while the dependence $\omega = \omega(N)$ for these excitations is situated above the lower boundary of "nonlinear gap." In the region $1.75 < \gamma < 2.5$ there exists the window of the oscillatory instability of the soliton solution, but for $\gamma > 2.5$ the gap and out-gap solitons are stable for all the frequencies.

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