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Exact Description of the Discrete Breathers and Solitons Interaction in the Nonlinear Transmission Lines

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For the nonlinear self-dual network equations and the equivalent Hirota lattice equation the pair collision processes of two discrete breathers, breather and one-parametric soliton (kink, antikink), breather and linear wave, one-parametric soliton and linear wave are described. The explicit expressions of the "kink-breather" and "breather-breather" solutions are constructed. The shifts of the center-of-masses and phases of the breather oscillations have been expressed in terms of the dynamical characteristics of linear and nonlinear excitations of the system.

1. Introduction

The dynamics of strongly excited low-dimensional lattices is described by nonlinear equations, but only a few of them can be solved exactly using the soliton theory. The most famous integrable discrete models are the Toda lattice[1] and the Ablowitz–Ladik system.[2] Multi-soliton solution of the Toda lattice equation has been found by Hirota.[3] The Toda lattice soliton is the pulse with the supersonic velocity (shock wave). Ablowitz and Ladik[2] have solved the integrable version of the discrete Nonlinear Schrödinger equation which is called the Ablowitz–Ladik (AL) equation using the inverse scattering method. In contrast to a pulse soliton of the Toda equation the AL soliton is two-parametric spatially localized and time periodic solution, i.e., it is a discrete version of breather-like excitation.[4] Hirota[5] has considered the exactly integrable system of the nonlinear lumped self-dual network (NLSDN) equations and has found its multi-soliton solutions. These equations describe the propagation of electrical signals in a cascade of four-terminal nonlinear LC self-dual circuits with the nonlinear dependence of the capacitance on the voltage and of the inductance on the current, respectively. The NLSDN equations are equivalent to the discrete modified Korteweg–de Vries (DMKdV) equation. Hirota has shown[5] that in the weakly nonlinear and continuum limit NLSDN and DMKdV equations reduce to the modified Kortewg–de Vries (MKdV) equation.

In the last decade the nonlinear solitonic transmission lines are investigated intensively.[6] Solitons are used in the optical and electrical transmission lines for high compacting of the signals in processes of information transmission and data storage. A great variety of the electrical solitonic transmission lines and devices, providing the generation, filtering and amplification of the solitonic pulses have been produced. Ricketts et al.[7] have introduced the electronic device generating a periodic stable train of electrical solitons — electrical soliton oscillator. The relative ease of the electronic transmission lines manufacture gives them many advantages over the photonic devices, involving light waves. Thus the exact analytical investigation of propagation and interaction of the discrete pulse solitons and breathers in the electrical transmission line described by the exactly integrable equations is the actual problem of mathematical physics because the results of its solving can be applicable in the data storage and transmission engineering.

Recently[8] the exact discrete breather solutions for the NLSDN equations have been found for the first time. The discrete breather is the self-localized oscillation which describes the intrinsic localized mode in a highly-localized high-frequency limit[9] and a dynamical bound state of the soliton-antisoliton pair in the case of the small values of the quasiwave number. Nimmo[10] has obtained and verified the multi-soliton solution of the NLSDN equations using wronskian technique. Zhou et al. received breather solution of the NLSDN equations using the wronskian technique.[11] In Ref. 12 the Hamiltonian dynamics of kinks and breathers is investigated for the DMKdV and MKdV equations. These nonlinear excitations are analogues of the shock waves and self-localized oscillations of the Fermi–Pasta–Ulam model which describes an anharmonic one-dimensional crystal. The dependence of energy on the momentum for kinks and quasiclassical spectra of energy for both continual and discrete breathers are found. In Ref. 13 the new classes of periodic solutions of the NLSDN equations describing the breather and soliton lattices expressed in terms of the Jacobi elliptic functions have been obtained.

It should be noticed that the Toda lattice model has exact solutions in the form of one-parametric pulse solitons but has no discrete breather solutions. The Ablowitz–Ladik model has a discrete breather-like solution but has no one-parametric pulse solitons. The remarkable feature of the NLSDN equations which are considered in this article is that it is the physically-important exactly-integrable discrete model which has simultaneously one-parametric kinks, antikinks and discrete breathers solutions. This makes it possible within a given integrable model of 1D lattice system to study analytically the processes of interaction between the discrete breathers as well as with the one-parametric solitons.

It is known that the interaction of the arbitrary number of solitons can be described by the sequence of their pair collisions. Currie[14] has extended the method of soliton collision analysis for the breather solutions in continuous systems. Thus it became possible to study the interaction of breathers with each other and with the other excitations of the system (kinks and linear waves). In Refs. 14 and 15 the interaction of breathers, kinks and linear waves is investigated for the sine-Gordon equation. In particular, using the "kink-breather" solution it is possible to derive the so-called wobbling kink or wobble solution. The wobble corresponds to a kink and a breather sitting on top of each other and not having a relative translational motion. In Ref. 16 the wobbling kink of the sine-Gordon equation is studied in detail. However up to the last year a corresponding analysis of breather and soliton interaction for discrete integrable models has been absent. Only very recently in Ref. 17 the discrete wobbling kink solution for the NLSDN equations have been obtained for the first time.

The problem of existence of an analogue of the sine-Gordon breather in near integrable systems describing, in particular, discrete models still remains of great interest.[4] Many efforts to obtain a stable sine-Gordon breather-like excitation have failed because of the reason that there are resonances in the dynamics of a localized excitation with the spectrum of plane waves of the system. These resonances lead to a radiation of energy out of the core of a local excitation and to its temporal decay. It is well known that all allowed plane wave frequencies fill a frequency range which is called a linear spectrum. Most spatially continuous field equations have the unbounded linear spectra. That makes resonances of higher order harmonics of a localized excitation with the linear spectrum unavoidable.

In the lattice systems with the acoustic linear spectrum the cyclic frequency of linear excitations starts from zero, depends periodically on the quasi-wave number and its absolute value has always a finite upper bound. The discrete breather has the main frequency and all its higher multiple harmonics lying above the linear spectrum hence this localized excitation will be dynamically stable. Due to the stability there is a huge advantage for experimental physics and engineering in the study of the breathers in discrete systems, rather than continuous.[4]

Thus the existence of the multi-soliton solution and the explicit expressions for the moving discrete breathers of NLSDN equations allows to describe exactly all interaction processes of the excitations. In integrable equations solitons keep their individuality, and their interaction results in the center-of-mass shifts in the case of kinks and additionally to the phase shifts of time oscillations in the case of breathers and linear waves. In the present paper we investigate the pair collision processes of the breathers with each other as well as with one-parametric solitons (kinks, antikinks) and linear waves. We discuss special cases: the interaction of a antikink and standing breather; the interaction of a kink and a linear wave; the interaction of moving and standing discrete breather, the interaction of discrete breather and a linear wave, as well as the linear superposition of two linear waves.

2. Hirota Lattice Model

In Ref. 5 Hirota has considered the nonlinear lumped self-dual network equations:

$$\frac{1}{1+V_n^2(t)}\frac{dV_n(t)}{dt} = I_n(t) - I_{n+1}(t),$$
$$\frac{1}{1+I_n^2(t)}\frac{dI_n(t)}{dt} = V_{n-1}(t) - V_n(t), \tag{1}$$

which describe dependence of voltage and current in the n-th capacitance and inductance, and are considered to be a discrete version of the MKDV equation.

By introducing the function $\phi_n(t)$:

$$V_n(t) \equiv \frac{d}{dt}\phi_{n+1/2}(t), \quad I_n(t) \equiv \frac{d}{dt}\phi_n(t) \tag{2}$$

Hirota[5] has transformed Eq. (1) into the form:

$$\frac{d}{dt}\phi_n(t) = \tan(\phi_{n-1/2} - \phi_{n+1/2}) \tag{3}$$

and derived for $\phi_n(t)$ the following equation, which is equivalent to Eq. (3):

$$\frac{\ddot{\phi}_n}{1+\dot{\phi}_n^2} = \tan(\phi_{n-1} - \phi_n) - \tan(\phi_n - \phi_{n+1}), \tag{4}$$

where dot means the derivative on time. This equation can be used to describe the dynamics of the 1D crystal, the Hirota lattice, with the interaction force between the nearest atoms proportional to the tangent of difference of their displacements:[8,12]

$$\frac{m\ddot{u}_n}{1+\dfrac{\pi^2}{4}\dfrac{\dot{u}_n^2}{s^2}} = \frac{2\gamma d_0}{\pi}\tan\left[\frac{\pi}{2}\left(\frac{u_{n-1}-u_n}{d_0}\right)\right] - \frac{2\gamma d_0}{\pi}\tan\left[\frac{\pi}{2}\left(\frac{u_n-u_{n+1}}{d_0}\right)\right], \tag{5}$$

where ϕ_n is the displacement of the n-th atom of mass m, $s = d_0\sqrt{\gamma/m}$ is the velocity of sound, d_0 is the lattice constant, γ is the force constant.

In dimensionless variables: $u_n/(2d_0/\pi) = \phi_n$, $\tau/(d_0/s) = t$ Eq. (5) reduces to Eq. (4).

Equation (4) can be derived from the Lagrangian:[8]

$$L = \sum_{n=-\infty}^{+\infty}\left\{\dot{\phi}_n \arctan\dot{\phi}_n - \frac{1}{2}\ln(1+\dot{\phi}_n^2) + \frac{1}{2}\ln[1+\tan^2(\phi_{n-1}-\phi_n)]\right\}. \tag{6}$$

Generalized momentum is found using Eq. (6):[8]

$$p_n = \frac{\partial L}{\partial\dot{\phi}_n} = \arctan\dot{\phi}_n. \tag{7}$$

The Hamilton function of the Eq. (4) has the form:[8]

$$H = \sum_{n=-\infty}^{+\infty}\left\{\frac{1}{2}\ln(1+\tan^2 p_n) + \frac{1}{2}\ln[1+\tan^2(\phi_{n-1}-\phi_n)]\right\}. \tag{8}$$

N-soliton solution of the Eq. (4) has the form:[5]

$$\phi_n^{(N)}(t) = \arctan\frac{g_n^{(N)}(t)}{f_n^{(N)}(t)}, \tag{9}$$

$$f_n^{(N)}(t) = \sum_{\mu=0,1}^{(e)}\exp\left(\sum_{i<j}^{N} B_{ij}\mu_i\mu_j + \sum_{j=1}^{N}\mu_j\eta_j\right),$$

$$g_n^{(N)}(t) = \sum_{\mu=0,1}^{(o)}\exp\left(\sum_{i<j}^{N} B_{ij}\mu_i\mu_j + \sum_{j=1}^{N}\mu_j\eta_j\right), \tag{10}$$

$$\eta_j = K_j n - \beta_j t + \eta_j^{(0)} = K_j(n - X_j(t)),$$
$$\beta_j = \varepsilon_j 2\sinh\left(\frac{K_j}{2}\right),$$
$$\varepsilon_j = \pm 1. \quad (j = 1, \ldots, N), \tag{11}$$

$$a_{ij} \equiv \exp(B_{ij}) = \frac{(\beta_i-\beta_j)^2 - 4\sinh^2[(K_i-K_j)/2]}{(\beta_i+\beta_j)^2 - 4\sinh^2[(K_i+K_j)/2]} = -\frac{\cosh[(K_i-K_j)/2] - \varepsilon_i\varepsilon_j}{\cosh[(K_i+K_j)/2] - \varepsilon_i\varepsilon_j}. \tag{12}$$

From (12) it is seen that $a_{ij} = -|a_{ij}| \leq 0$, $|a_{ij}| > 1$ if $K_iK_j < 0$ and $|a_{ij}| < 1$ if $K_iK_j > 0$. Coefficients $a_{j_1 \ldots j_m} \equiv a(j_1, \ldots, j_m)$ are given by the formulas:

$$a(j_1, \ldots, j_m) = \begin{cases} \prod_{k<l}^{(m)} a(j_k, j_l), & m > 2, \\ 1, & m = 0, 1. \end{cases} \tag{13}$$

By $\sum_{\mu=0,1}^{(e,o)}$ a sum over all sets $(\mu_1, \ldots, \mu_N)$ is defined, each μ_j is 0 or 1, and $\sum_{j=1}^{N}{}^{(e,o)}\mu_j$ is an (even, odd) integer, respectively. Parameters ε_j can take the values -1 or $+1$ and correspond to the sign of the solitons velocities. Coefficients a_{ij} define the phase shifts of solitons during their collisions. In the case of one-parametric solitons the shift ΔX_i of the center-of-mass position ($X_i(t) = V_i t + X_{i0}$):

$$X_i(t \to +\infty) = X_i(t \to -\infty) + \Delta X_i \tag{14}$$

for the i-th soliton equals

$$\Delta X_i = -\,\mathrm{sign}[K_j(V_i - V_j)]\,\frac{\ln |a_{ij}|}{K_i}. \tag{15}$$

As it is seen from the Eq. (12), the shifts of two solitons will be different depending on whether the solitons move in one direction or towards each other. The initial phases are denoted by $\eta_j^{(0)} = K_j X_{j0}$. Hirota's solution is valid for the infinite interval $n \in (-\infty, +\infty)$.

One-soliton solution (one-parametric soliton solution) of the Eq. (4) is the kink. In mechanical model one-parametric soliton corresponds to the supersonic pulse of compression or stretching, in the model of the NLSDN equations one-parametric soliton corresponds to the electrical pulse of the current strength or voltage.

$$\phi_n^{(1)} = \arctan[\exp(\eta_1)], \tag{16}$$

$$\eta_1 = K_1(n - V_1 t - X_{10}) = K_1(n - X_1(t)), \tag{17}$$

where $X_1(t) = V_1 t + X_{10}$ is the position of the one-parametric soliton center-of-mass, $\eta_1^{(0)} = K_1 X_{10}$ is the arbitrary constant. In the following we set this constant equal to zero for simplicity.

Kink (antikink in the case $K_1 < 0$) is the nonlinear pulse moving along the lattice with the velocity $V_1 \geq 1$:

$$V_1 = \varepsilon_1 \frac{\sinh(K_1/2)}{K_1/2}. \tag{18}$$

Parameter $\Delta_1 \sim 1/K_1$ is the effective width of kink. Parameter $\varepsilon_1 = \pm 1$ defines the sign of the kink velocity.

Two-soliton solution is given as

$$\phi_n^{(2)} = \arctan\left[\frac{\exp(\eta_1) + \exp(\eta_2)}{1 + a_{12}\exp(\eta_1)\exp(\eta_2)}\right]. \tag{19}$$

Parameters $\eta_j, \beta_j, \varepsilon_j, a_{ij}, (j = 1, 2)$ are defined according to Eqs. (11) and (12).

In Ref. 8 the exact discrete breather solution has been found. Discrete breather is the localized oscillation travelling along the lattice. To get breather solution one has to take the complex conjugative parameters of the two-soliton solution. The breather solution has the form:

$$\phi_n^{(b)} = \arctan\left[\frac{\sinh(\kappa/2)}{\sin(k/2)}\,\frac{\cos(kn - \omega t + \Phi_0)}{\cosh \kappa(n - Vt - X_0)}\right], \tag{20}$$

$$V = \varepsilon\,\frac{\sinh(\kappa/2)\cos(k/2)}{\kappa/2},$$

$$V_{\mathrm{ph}} = \frac{\omega}{k} = \varepsilon\,\frac{\cosh(\kappa/2)\sin(k/2)}{k/2}, \tag{21}$$

where k is the quasi-wave number, $\varepsilon = \pm 1$. The amplitude of the breather equals $A^{(b)} = \arctan[\sinh(\kappa/2)/\sin(k/2)]$ and the parameter $\Delta \sim 1/\kappa$ is the effective width of the breather. Parameters Φ_0, X_0 are the arbitrary constants.

Unlike sine-Gordon breather the discrete breather (20) of the NLSDN equations is a localized anti-phase oscillation. The frequency of the discrete breather (20) lies above the spectrum band of linear waves while the frequency of the sine-Gordon breather lies below the lowest edge of the optical-like spectrum of linear excitations.

Expressions (21) represent the center-of-mass velocity and phase velocity of the carrier wave respectively. By ω we denote the frequency of breather oscillations. From the expressions (21) it is seen that the carrier wave can propagate either in the same or opposite direction with the center-of-mass (envelope).

$$V \uparrow\uparrow V_{\mathrm{ph}} \Leftrightarrow k = [0 + 2\pi l, \pi + 2\pi l], \quad (l = 0, 1, 2, \ldots), \tag{22}$$

$$V \uparrow\downarrow V_{\mathrm{ph}} \Leftrightarrow k = [\pi + 2\pi l, 2\pi + 2\pi l], \quad (l = 0, 1, 2, \ldots). \tag{23}$$

The center-of-mass velocity can take any values.

In Fig. 1 the standing discrete breathers with different amplitude (effective width) are shown. Breather in Fig. 1(a) has large amplitude and is highly localized (it is localized approximately on the five lattice sites). Breather in Fig. 1(b) has small amplitude and is weakly localized.

3. Collision of Breather and One-Parametric Soliton

Using the three-soliton solution $\phi_n^{(3)}$ the interaction of breather with one-parametric soliton (kink or antikink) was described. To get the case "kink-breather" one has to bound two solitons in breather in this solution.

$$\phi_n^{(3)} = \arctan\left[\frac{\exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + a_{123}\exp(\eta_1)\exp(\eta_2)\exp(\eta_3)}{1 + a_{12}\exp(\eta_1)\exp(\eta_2) + a_{13}\exp(\eta_1)\exp(\eta_3) + a_{23}\exp(\eta_2)\exp(\eta_3)}\right], \tag{24}$$

$$\eta_1 = K_1 n - \beta_1 t + \eta_1^{(0)}, \quad \eta_2 = \eta_3^* = \Theta + i\vartheta, \tag{25}$$

$$K_2 = K_3^* = \kappa + ik, \quad \beta_2 = \beta_3^* = \Omega + i\omega, \tag{26}$$

$$\vartheta = kn - \omega t + \Phi_0, \quad \Theta = \kappa(n - X_0) - \Omega t, \tag{27}$$

$$\beta_1 = \varepsilon_1 2\sinh\frac{K_1}{2}, \quad \Omega = \varepsilon 2\sinh\frac{\kappa}{2}\cos\frac{k}{2}, \quad \omega = \varepsilon 2\cosh\frac{\kappa}{2}\sin\frac{k}{2}, \tag{28}$$

$$\varepsilon_1 = \pm 1, \quad \varepsilon_2 = \varepsilon_3 = \varepsilon = \pm 1, \tag{29}$$

$$a_{12} = a_{13}^* = -\frac{\cosh[(K_1 - K_2)/2] - \varepsilon_1\varepsilon_2}{\cosh[(K_1 + K_2)/2] - \varepsilon_1\varepsilon_2} \equiv R\exp(i\delta), \quad a_{23} = \frac{\sin^2(k/2)}{\sinh^2(\kappa/2)}, \tag{30}$$

$$a_{123} = a_{12}a_{13}a_{23}. \tag{31}$$

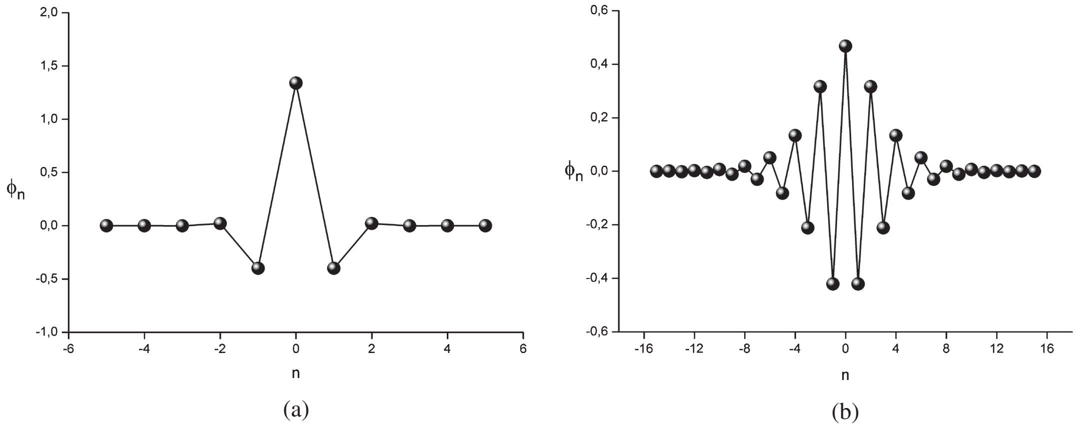

Fig. 1. The discrete breather solutions of the Hirota lattice equation and equivalent NLSDN equations. $t = 0.0$, $k = \pi$ (standing breathers). Balls indicate the displacements of atoms. Lines connecting balls are guides for the eye. (a) $\kappa = 3.0$ (highly localized breather) (b) $\kappa = 0.5$ (small amplitude breather).

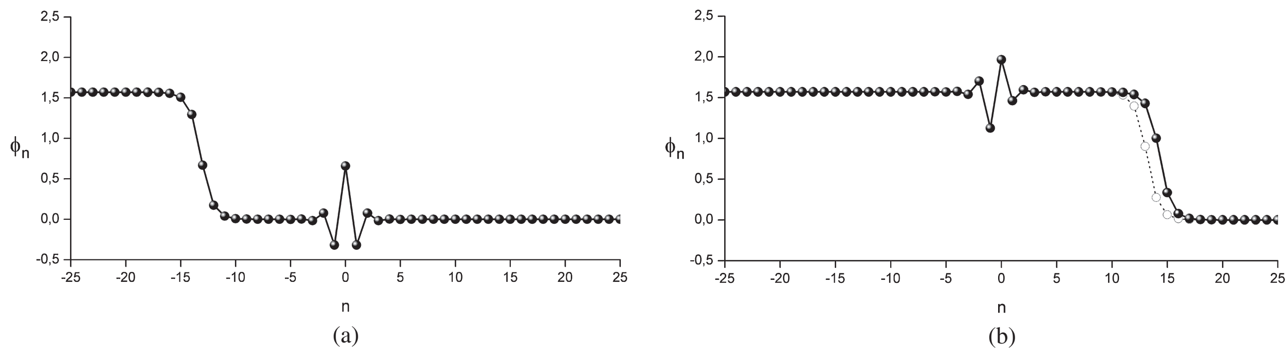

Fig. 2. The process of antikink and standing breather collision $\varepsilon_1 = +1$, $\varepsilon = +1$, $K_1 = -1.5$, $\kappa = 1.5$, $k = \pi$. (a) Before the collision $t = -12.0$, (b) after the collision $t = +12.0$, dotted line corresponds to the antikink profile if there would be no breather.

Parameters $\eta_1^{(0)}, \Phi_0, X_0$ are the arbitrary constants.

Solitons 2 and 3 we have bound in breather by choosing the complex conjugative parameters. As a result we get the "kink-breather" solution that describes the lattice, containing one-parametric soliton and breather, propagating with the arbitrary velocities.

$$\phi_n^{(kb)} = \arctan\left[\frac{\exp(\eta_1) + \dfrac{1}{R\sqrt{a_{23}}}\dfrac{\cos\vartheta}{\cosh\Theta}}{\dfrac{1}{R}\dfrac{\cosh(\Theta - \ln R)}{\cosh\Theta} + \exp(\eta_1)\dfrac{1}{\sqrt{a_{23}}}\dfrac{\cos(\vartheta+\delta)}{\cosh\Theta}}\right]. \tag{32}$$

Let us consider the process of collision of one-parametric soliton and breather. For definiteness we investigated the situation when breather is standing ($\kappa > 0, k = \pi$), and antikink ($K_1 < 0$) moves along the n-axis towards breather ($\varepsilon_1 = +1$). The asymptotes of the solution (32) for $t \to \pm\infty$ are:

$$\begin{aligned}\phi_n^{(kb)}(t \to -\infty) &= \arctan[\exp(\eta_1)] \\ &\quad + \arctan\left[\frac{1}{\sqrt{a_{23}}}\frac{\cos\vartheta}{\cosh(\Theta - \ln R)}\right], \\ \phi_n^{(kb)}(t \to +\infty) &= \arctan[\exp(\eta_1 + 2\ln R)] \\ &\quad + \frac{\pi}{2} - \arctan\left[\frac{1}{\sqrt{a_{23}}}\frac{\cos(\vartheta+\delta)}{\cosh\Theta}\right].\end{aligned} \tag{33}$$

In Fig. 2 the process of antikink and standing breather collision is shown.

For $t \to -\infty$ we have separated antikink and breather [Fig. 2(a)], antikink approaches to the standing breather. After that the collision process is performed, during which antikink passes though the breather. For $t \to +\infty$ antikink moves away from standing breather [Fig. 2(b)]. Antikink and breather recover their individuality and velocity. Dotted line in the Fig. 2(b) corresponds to the antikink profile if there would be no breather. The difference in the asymptotes of the solution (32) before and after the collision is the center-of-mass shifts of antikink and breather ($\Delta X_1, \Delta X_b$) and the shift of the breather phase of oscillations ($\Delta\Phi$):

$$X_1(t \to +\infty) = X_1(t \to -\infty) + \Delta X_1 \tag{34}$$

$$X_b(t \to +\infty) = X_b(t \to -\infty) + \Delta X_b \tag{35}$$

$$\Phi(t \to +\infty) = \Phi(t \to -\infty) + \Delta\Phi \tag{36}$$

$$\Delta X_1 = -\frac{2\ln R}{K_1} = -\frac{2}{K_1}\ln\left(\frac{\cosh\dfrac{K_1 - \kappa}{2}}{\cosh\dfrac{K_1 + \kappa}{2}}\right) \geq 0,$$

$$\Delta X_b = -\frac{\ln R}{\kappa} = -\frac{1}{\kappa}\ln\left(\frac{\cosh\dfrac{K_1-\kappa}{2}}{\cosh\dfrac{K_1+\kappa}{2}}\right) \leq 0. \tag{37}$$

$$\Delta\Phi = \delta + \pi = -2\arctan\left(\frac{\cosh\kappa/2}{\sinh K_1/2}\right) + \pi. \tag{38}$$

In the considered situation when breather stands and antikink propagates along the n-axis towards breather, the breather center-of-mass shifts to the negative direction of the n-axis, the antikink center-of-mass shifts towards the positive direction of the n-axis. The interaction of antikink and breather has the character of the effective attraction.

In the general case for the arbitrary values of the parameters of the "kink-breather" solution the center-of-mass shifts of one-parametric soliton and breather and the shift of breather phase of oscillations are given by the formulas:

$$\Delta X_1 = -\,\mathrm{sign}[\kappa(V_1 - V_b)]\,\frac{2\ln R}{K_1},$$

$$\Delta X_b = -\,\mathrm{sign}[K_1(V_b - V_1)]\,\frac{\ln R}{\kappa}. \tag{39}$$

$$\Delta\Phi = +\,\mathrm{sign}[K_1(V_b - V_1)]\delta + \pi. \tag{40}$$

$$R = \frac{\cosh\dfrac{K_1-\kappa}{2} - \varepsilon_1\varepsilon\cos\dfrac{k}{2}}{\cosh\dfrac{K_1+\kappa}{2} - \varepsilon_1\varepsilon\cos\dfrac{k}{2}},$$

$$\delta = 2\arctan\left(\frac{\cosh\dfrac{K_1}{2}\cos\dfrac{k}{2} - \varepsilon_1\varepsilon\cosh\dfrac{\kappa}{2}}{\sinh\dfrac{K_1}{2}\sin\dfrac{k}{2}}\right). \tag{41}$$

Obviously the quantity R is always greater than or equal to zero for any values of the parameters $K_1, \kappa, k, \varepsilon_1, \varepsilon$, and

$$\begin{cases} 0 < R < 1 \Leftrightarrow K_1\kappa > 0, \\ R > 1 \Leftrightarrow K_1\kappa < 0. \end{cases} \tag{42}$$

The quantity $R \to 1$ when the amplitude of the kink or breather tends to zero ($K_1 \to 0$ or $\kappa \to 0$). Parameter δ varies in the interval $[-\pi + 2\pi l, \pi + 2\pi l]$, $l \in \mathbb{Z}$.

The center-of-mass shifts correspond to the mutual attraction of the one-parametric solitons (kinks, antikinks) and breathers. From Eq. (39) it is seen that

$$\left|\frac{\Delta X_1}{\Delta X_b}\right| = 2\left|\frac{\kappa}{K_1}\right|. \tag{43}$$

In Fig. 3 the scattering data of the collision process of the standing breather and kink are shown.

From the "kink-breather" solution, the wobbling kink solution (Fig. 4) has been obtained.[17] The wobbling kink is a nonlinear superposition of the kink and breather, the center-of-mass positions and velocities of which are equal.

4. Collision of Two Breathers

Using the four-soliton solution $\phi_n^{(4)}$

$$\phi_n^{(4)} = \arctan\frac{g_n^{(4)}(t)}{f_n^{(4)}(t)}, \tag{44}$$

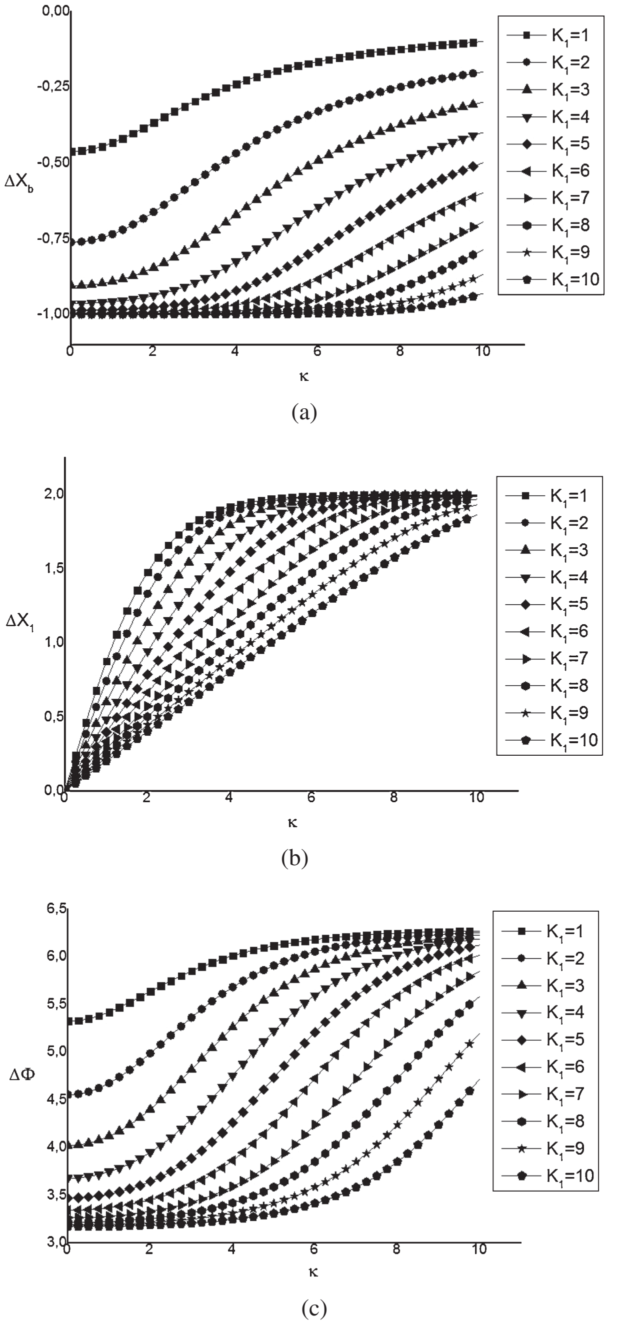

Fig. 3. The scattering data of the collision process of the standing breather and kink. The center-of-mass shifts of the standing breather (a) and one-parametric soliton (b) and the shift of breather phase of oscillations (c) as the functions of the parameters K_1, κ of the "kink-breather" solution. $K_1 \geq 0$, $\kappa \geq 0$, $k = \pi$, $\varepsilon_1 = +1$, $\varepsilon = +1$.

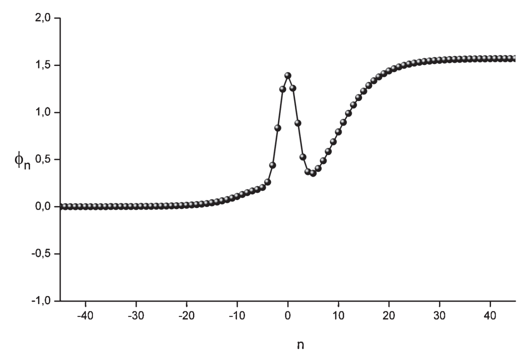

Fig. 4. The wobbling kink solution of the Hirota lattice equation and equivalent NLSDN equations. $t = 1.0$, $\varepsilon_1 = +1$, $\varepsilon = +1$, $K_1 = 0.2$, $\kappa = 0.7$, $k = 0.39$.

$$g_n^{(4)}(t) = e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_4} + a_{234}e^{\eta_2+\eta_3+\eta_4} + a_{134}e^{\eta_1+\eta_3+\eta_4} + a_{124}e^{\eta_1+\eta_2+\eta_4} + a_{123}e^{\eta_1+\eta_2+\eta_3},$$

$$f_n^{(4)}(t) = 1 + a_{12}e^{\eta_1+\eta_2} + a_{13}e^{\eta_1+\eta_3} + a_{14}e^{\eta_1+\eta_4} + a_{23}e^{\eta_2+\eta_3} + a_{24}e^{\eta_2+\eta_4} + a_{34}e^{\eta_3+\eta_4} + a_{1234}e^{\eta_1+\eta_2+\eta_3+\eta_4}. \quad (45)$$

The interaction of two breathers, breather and linear wave and two linear waves with each other was described. For this purpose, as in the previous case of three-soliton solution, one has to "bound" one-parametric solitons in breathers by choosing the complex conjugative parameters. Solitons with numbers 1 and 3 we have "bound" in one breather and assigned it by the index 1, while the solitons with numbers 2 and 4 we have "bound" in another breather and assigned it by the index 2.

$$K_1 = K_3^* \equiv \kappa_1 + ik_1, \quad K_2 = K_4^* \equiv \kappa_2 + ik_2, \quad (46)$$

$$\eta_1 = \eta_3^* = \Theta_1 + i\vartheta_1, \quad \eta_2 = \eta_4^* = \Theta_2 + i\vartheta_2, \quad (47)$$

$$\Theta_j = \kappa_j(n - X_{j0}) - \Omega_j t, \quad \vartheta_j = k_j n - \omega_j t + \Phi_{j0}, \quad (j = 1, 2), \quad (48)$$

$$\Omega_j = \varepsilon_j 2\sinh\frac{\kappa_j}{2}\cos\frac{k_j}{2}, \quad \omega_j = \varepsilon_j 2\cosh\frac{\kappa_j}{2}\sin\frac{k_j}{2}, \quad (j = 1, 2), \quad (49)$$

$$\varepsilon_3 = \varepsilon_1 = \pm 1, \quad \varepsilon_4 = \varepsilon_2 = \pm 1, \quad (50)$$

$$a_{13} = \frac{\sin^2(k_1/2)}{\sinh^2(\kappa_1/2)}, \quad a_{24} = \frac{\sin^2(k_2/2)}{\sinh^2(\kappa_2/2)}, \quad (51)$$

$$a_{12} = a_{34}^* = -\frac{\cosh\dfrac{K_1 - K_2}{2} - \varepsilon_1\varepsilon_2}{\cosh\dfrac{K_1 + K_2}{2} - \varepsilon_1\varepsilon_2} \equiv R_{12}\exp(i\delta_{12}),$$

$$a_{14} = a_{23}^* = -\frac{\cosh\dfrac{K_1 - K_4}{2} - \varepsilon_1\varepsilon_2}{\cosh\dfrac{K_1 + K_4}{2} - \varepsilon_1\varepsilon_2} \equiv R_{14}\exp(i\delta_{14}). \quad (52)$$

Parameters $\Phi_{10}, X_{10}, \Phi_{20}, X_{20}$ are the arbitrary constants. As a result we get the "breather-breather" solution.

$$\phi_n^{(bb)} = \arctan\frac{g_n^{(bb)}(t)}{f_n^{(bb)}(t)}, \quad (53)$$

$$g_n^{(bb)} = \frac{1}{\sqrt{a_{13}}}\exp(-\Theta_2)\cos\left(\vartheta_1 - \frac{b_1}{2}\right) + \frac{1}{\sqrt{a_{24}}}\exp(-\Theta_1)\cos\left(\vartheta_2 - \frac{b_2}{2}\right) + \frac{1}{\sqrt{a_{13}}}\exp(+\Theta_2)\cos\left(\vartheta_1 + \frac{b_1}{2}\right) + \frac{1}{\sqrt{a_{24}}}\exp(+\Theta_1)\cos\left(\vartheta_2 + \frac{b_2}{2}\right), \quad (54)$$

$$f_n^{(bb)} = \sqrt{R_{12}R_{14}}\cosh(\Theta_1 + \Theta_2) + \frac{1}{\sqrt{R_{12}R_{14}}}\cosh(\Theta_1 - \Theta_2) + \sqrt{\frac{R_{12}}{R_{14}}}\frac{1}{\sqrt{a_{13}a_{24}}}\cos(\vartheta_1 + \vartheta_2) + \sqrt{\frac{R_{14}}{R_{12}}}\frac{1}{\sqrt{a_{13}a_{24}}}\cos(\vartheta_1 - \vartheta_2), \quad (55)$$

$$b_1 = \delta_{12} + \delta_{14} \equiv \delta_1, \quad b_2 = \delta_{12} - \delta_{14} \equiv 2\pi + \delta_2, \quad (56)$$

$$\delta_i = -2\arctan\frac{F_{ij}}{G_{ij}}, \quad (57)$$

where

$$F_{ij} = 2\sinh\frac{\kappa_j}{2}\sin\frac{k_i}{2}\left(\cosh\frac{\kappa_j}{2}\cos\frac{k_i}{2} - \varepsilon_i\varepsilon_j\cosh\frac{\kappa_i}{2}\cos\frac{k_j}{2}\right), \quad (58)$$

$$G_{ij} = 2\cosh\frac{\kappa_j}{2}\cos\frac{k_i}{2}\left(\cosh\frac{\kappa_j}{2}\cos\frac{k_i}{2} - \varepsilon_i\varepsilon_j\cosh\frac{\kappa_i}{2}\cos\frac{k_j}{2}\right) - \left(\cosh^2\frac{\kappa_j}{2} - \cosh^2\frac{\kappa_i}{2}\right) + \left(\cos^2\frac{k_j}{2} - \cos^2\frac{k_i}{2}\right), \quad (59)$$

where $i = 1, 2,\ j = 1, 2,\ i \neq j$.

Let us consider the process of collision of two breathers. For definiteness we took the case when the breather with index 2 is standing ($\kappa_2 > 0, k_2 = \pi$), and breather with index 1 ($\kappa_1 > 0, k_1 > 0$) moves along the n-axis ($\varepsilon_1 = +1$) towards breather 2. The asymptotes of the solution (47)–(51) for $t \to \pm\infty$ are:

$$\begin{cases} \phi_n^{(bb)}(t \to -\infty) = \arctan\left\{\dfrac{1}{\sqrt{a_{13}}}\dfrac{\cos\vartheta_1}{\cosh[\Theta_1 - \ln\sqrt{R_{12}R_{14}}]}\right\} + \arctan\left\{\dfrac{1}{\sqrt{a_{24}}}\dfrac{\cos(\vartheta_2 + b_2)}{\cosh[\Theta_2 + \ln\sqrt{R_{12}R_{14}}]}\right\}, \\ \phi_n^{(bb)}(t \to +\infty) = \arctan\left\{\dfrac{1}{\sqrt{a_{13}}}\dfrac{\cos(\vartheta_1 + b_1)}{\cosh[\Theta_1 + \ln\sqrt{R_{12}R_{14}}]}\right\} + \arctan\left\{\dfrac{1}{\sqrt{a_{24}}}\dfrac{\cos\vartheta_2}{\cosh[\Theta_2 - \ln\sqrt{R_{12}R_{14}}]}\right\}. \end{cases} \quad (60)$$

In Fig. 5 the process of collision of moving and standing breathers is shown.

For $t \to -\infty$ we have two separated breathers [Fig. 5(a)], breather with index 1 approaches to the standing breather with index 2. After that the collision process is performed, during which breather 1 passes though the breather 2 [Fig. 5(b)]. For $t \to +\infty$ breather 1 moves away from breather 2. Both breathers recover their individuality and velocity. The difference before and after the collision is the center-of-mass shifts ($\Delta X_1, \Delta X_2$) and the shifts of the phases of oscillations ($\Delta\Phi_1, \Delta\Phi_2$) [Fig. 5(c)]:

$$\Delta X_1 = -\frac{1}{\kappa_1}\ln\left[\frac{\cosh^2\left(\dfrac{\kappa_1 - \kappa_2}{2}\right) - \sin^2\dfrac{k_1}{2}}{\cosh^2\left(\dfrac{\kappa_1 + \kappa_2}{2}\right) - \sin^2\dfrac{k_1}{2}}\right] \geq 0,$$

$$\Delta X_2 = +\frac{1}{\kappa_2}\ln\left[\frac{\cosh^2\left(\dfrac{\kappa_1 - \kappa_2}{2}\right) - \sin^2\dfrac{k_1}{2}}{\cosh^2\left(\dfrac{\kappa_1 + \kappa_2}{2}\right) - \sin^2\dfrac{k_1}{2}}\right] \leq 0, \quad (61)$$

$$\Delta\Phi_1 = b_1, \quad \Delta\Phi_2 = -b_2, \quad (62)$$

where the quantities b_1, b_2 are defined by Eqs. (56)–(59).

In the considered situation when breather 2 stands and breather 1 propagates along the n-axis towards breather 2,

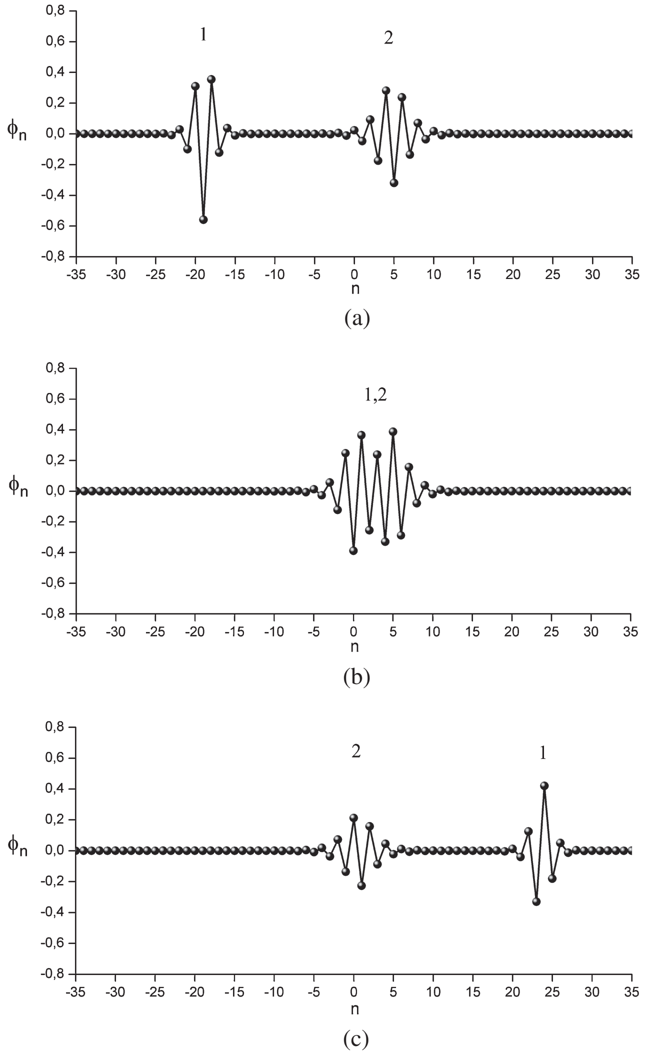

Fig. 5. The process of collision of moving (number 1) and standing (number 2) breathers, $\varepsilon_1 = +1$, $\varepsilon_2 = +1$, $\kappa_1 = 1.2$, $\kappa_2 = 0.7$, $k_1 = 3.04$, $k_2 = \pi$: (a) before the collision $t = -380.0$, (b) during the collision $t = 0.0$, and (c) after the collision $t = +380.0$.

the breather 2 center-of-mass shifts to the negative direction of the n-axis, while the breather 1 center-of-mass shifts towards the positive direction of the n-axis. The interaction of breathers has the character of the effective attraction.

In the general case for the arbitrary values of the parameters of the "breather-breather" solution the breathers center-of-mass shifts and the phase of oscillations shifts are given by the formulas

$$\Delta X_1 = -\,\mathrm{sign}[\kappa_2(V_1 - V_2)]\,\frac{\ln R_{12}R_{14}}{\kappa_1}, \tag{63}$$

$$\Delta X_2 = -\,\mathrm{sign}[\kappa_1(V_2 - V_1)]\,\frac{\ln R_{12}R_{14}}{\kappa_2}, \tag{64}$$

$$\Delta\Phi_1 = +\,\mathrm{sign}[\kappa_2(V_1 - V_2)]b_1, \tag{65}$$

$$\Delta\Phi_2 = +\,\mathrm{sign}[\kappa_1(V_2 - V_1)]b_2, \tag{66}$$

where the quantities R_{12}, R_{14} equal:

$$R_{12} = \frac{\cosh\dfrac{\kappa_1 - \kappa_2}{2} - \varepsilon_1\varepsilon_2\cos\dfrac{k_1 - k_2}{2}}{\cosh\dfrac{\kappa_1 + \kappa_2}{2} - \varepsilon_1\varepsilon_2\cos\dfrac{k_1 + k_2}{2}},$$

$$R_{14} = \frac{\cosh\dfrac{\kappa_1 - \kappa_2}{2} - \varepsilon_1\varepsilon_2\cos\dfrac{k_1 + k_2}{2}}{\cosh\dfrac{\kappa_1 + \kappa_2}{2} - \varepsilon_1\varepsilon_2\cos\dfrac{k_1 - k_2}{2}} \tag{67}$$

and the quantities b_1, b_2 are defined by Eqs. (56)–(59).

The expressions for the center-of-mass shifts contain the product $R_{13}R_{14}$. It is easy to see that

$$\begin{cases} 0 < R_{12}R_{14} < 1 \Leftrightarrow \kappa_1\kappa_2 > 0, \\ R_{12}R_{14} > 1 \Leftrightarrow \kappa_1\kappa_2 < 0. \end{cases} \tag{68}$$

The product $R_{12}R_{14} \to 1$ when the amplitude of at least one of the breathers tends to zero ($\kappa_1 \to 0$ or $\kappa_2 \to 0$). Parameters δ_i vary in the interval $[-\pi + 2\pi l, \pi + 2\pi l]$, $l \in \mathbb{Z}$.

The center-of-mass shifts correspond to the mutual attraction of the breathers. From Eqs. (63)–(66) it is seen that

$$\left|\frac{\Delta X_1}{\Delta X_2}\right| = \left|\frac{\kappa_2}{\kappa_1}\right|, \tag{69}$$

$$|\Delta\Phi_1| = |\Delta\Phi_2|. \tag{70}$$

We have investigated the collision process of the two equal breathers (Fig. 6) with the following values of the parameters $\kappa_1 = \kappa_2 \equiv \kappa \geq 0$, $k_1 = k_2 \equiv k \geq 0$, $\varepsilon_1 = +1, \varepsilon_2 = -1$. In this case Eqs. (63)–(66) reduce to

$$\Delta X_1 = -\Delta X,\ \Delta X_2 = +\Delta X,$$
$$\Delta X = \frac{1}{\kappa}\ln\left[\frac{2(1 + \cos k)}{(1 + \cosh\kappa)(\cosh\kappa + \cos k)}\right]. \tag{71}$$

$$\Delta\Phi_1 = -\Delta\Phi,\ \Delta\Phi_2 = +\Delta\Phi,$$
$$\Delta\Phi = 2\arctan\left(\tanh\frac{\kappa}{2}\tan\frac{k}{2}\right). \tag{72}$$

5. The Scattering of Linear Waves on the Kink and Breather

Let us describe the process of passing a linear wave through soliton. We seek a solution in the form:

$$\phi_n = \phi_n^{(s)} + \phi_n^{(L)}, \quad \phi_n^{(L)} \ll \phi_n^{(s)}, \tag{73}$$

where $\phi_n^{(s)}$ is a multi-soliton solution, $\phi_n^{(L)}$ is a linear wave.

Using the fact that Eqs. (3) and (4) are equivalent we will analyze Eq. (3) instead of (4) because it is more simple. The solutions, which we get will be valid for both Eqs. (3) and (4).

After linearization in $\phi_n^{(L)}$, (3) is reduced to the equation:

$$\frac{d}{dt}\phi_n^{(L)} = (\phi_{n+1/2}^{(L)} - \phi_{n-1/2}^{(L)})[1 + \tan^2(\phi_{n+1/2}^{(s)} - \phi_{n-1/2}^{(s)})]. \tag{74}$$

Equation (74) describes the dynamics of linear wave at the presence of arbitrary number of solitons and breathers given by the multi-soliton solution $\phi_n^{(s)}$. Let us consider the process of passing a harmonic vibration through one soliton (kink or antikink). In this case Eq. (74) is reduced to:

$$\frac{d}{dt}\phi_n^{(L)} = (\phi_{n+1/2}^{(L)} - \phi_{n-1/2}^{(L)})\left\{1 + \frac{\sinh^2\dfrac{K_1}{2}}{\cosh^2[K_1(n - V_1 t)]}\right\}. \tag{75}$$

The solution of the Eq. (75) is given by

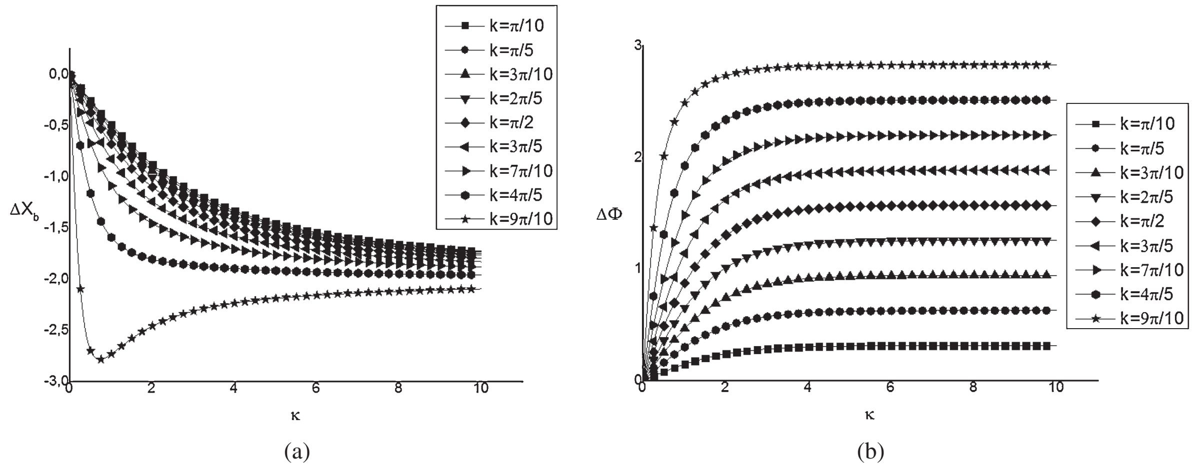

Fig. 6. The scattering data of the collision process of the two equal breathers. The modulus of the breather center-of-mass shifts (a) and the modulus of the shift of breather phase of oscillations (b) as the functions of the parameters κ, k of the "breather-breather" solution. $\kappa_1 = \kappa_2 \equiv \kappa \geq 0$, $k_1 = k_2 \equiv k \geq 0$, $\varepsilon_1 = +1$, $\varepsilon_2 = -1$.

$$\phi_n^{(\mathrm{L})} = -\zeta \cos\frac{\delta}{2}\left(-\tan\frac{\delta}{2} + i\tanh\eta_1\right)\exp(i\vartheta), \tag{76}$$

where η_1 is defined by Eq. (17) and

$$\vartheta = kn - \omega_{\mathrm{L}} t, \quad \omega_{\mathrm{L}} = 2\sin\frac{k}{2}, \tag{77}$$

$$\delta = 2\arctan\left(\frac{\cosh\frac{K_1}{2}\cos\frac{k}{2} - \varepsilon_1\varepsilon}{\sinh\frac{K_1}{2}\sin\frac{k}{2}}\right). \tag{78}$$

Equation (78) represent the shift of the linear wave oscillations.

The same result can be obtained using the "kink-breather" solution (32) when $\kappa = 0$. In this limit breather is delocalized and reduced to the linear wave. Expression (32) reduces to the form ($\phi_n^{(kb)} \to \phi_n^{(\mathrm{kL})}$):

$$\phi_n^{(\mathrm{kL})} = \arctan\left[\frac{\exp(\eta_1) + \zeta\cos(\vartheta)}{1 + \zeta\exp(\eta_1)\cos(\vartheta + \delta)}\right], \tag{79}$$

where $\zeta = (a_{23})^{-1/2} \ll 1$ is the small amplitude of the linear wave, $\vartheta = kn - \omega_{\mathrm{L}} t$, $\omega_{\mathrm{L}} = 2\sin(k/2)$ is the dispersion law for the linear waves. Expressions (39) and (40) are reduced in the following way:

$$\Delta X_1 = 0 = \Delta X_{\mathrm{L}},$$

$$\Delta\Phi_{\mathrm{L}} = +\operatorname{sign}[K_1(V_b - V_1)] \times 2\arctan\left(\frac{\cosh\frac{K_1}{2}\cos\frac{k}{2} - \varepsilon_1\varepsilon}{\sinh\frac{K_1}{2}\sin\frac{k}{2}}\right) + \pi. \tag{80}$$

From Eq. (80) it is seen that during the collision of kink and linear wave, kink does not undergo the center-of-mass shift, and the linear wave has the shift of the oscillations.

Using the special case of the "breather-breather" solution (53) when, for example, $\kappa_2 \to 0$ the interaction between breather and linear wave can be described. In this limit breather 2 is delocalized and reduced to the linear wave. Expression (53) reduces to the form ($\phi_n^{(bb)} \to \phi_n^{(\mathrm{bL})}$):

$$\phi_n^{(\mathrm{bL})} = \arctan\left[\frac{\frac{1}{\sqrt{a_{13}}}\frac{\cos\vartheta_1}{\cosh\Theta_1} + \frac{\exp(-\Theta_1)}{\cosh\Theta_1}\zeta\cos\left(\vartheta_2 - \frac{b_2}{2}\right) + \frac{1}{\sqrt{a_{13}}}\frac{\cos\vartheta_1}{\cosh\Theta_1} + \frac{\exp(+\Theta_1)}{\cosh\Theta_1}\zeta\cos\left(\vartheta_2 + \frac{b_2}{2}\right)}{2 + R_{12}\zeta\frac{1}{\sqrt{a_{13}}}\frac{\cos(\vartheta_1 + \vartheta_2)}{\cosh\Theta_1} + R_{14}\zeta\frac{1}{\sqrt{a_{13}}}\frac{\cos(\vartheta_1 - \vartheta_2)}{\cosh\Theta_1}}\right], \tag{81}$$

where $\zeta = (a_{24})^{-1/2} \ll 1$ is the small amplitude of the linear wave, $\vartheta = kn - \omega_{\mathrm{L}} t$, $\omega_{\mathrm{L}} = 2\sin(k/2)$ is the dispersion law for the linear wave. Expressions (63)–(66) are reduced in the following way:

$$\Delta X_1 = 0 = \Delta X_{\mathrm{L}}, \tag{82}$$

$$\Delta\Phi_1 = 0, \tag{83}$$

$$\Delta\Phi_2 = +\operatorname{sign}[\kappa_1(V_2 - V_1)] \times 4\arctan\left(\frac{\cosh\frac{\kappa_1}{2}\cos\frac{k_2}{2} - \varepsilon_1\varepsilon_2\cos\frac{k_1}{2}}{\sinh\frac{\kappa_1}{2}\sin\frac{k_2}{2}}\right). \tag{84}$$

From Eqs. (82)–(84) it is seen that during the collision of breather and linear wave, breather does not undergo either center-of-mass shift or shift of phase of oscillations, while the linear wave has the shift of the oscillations.

If the amplitudes of both breathers in Eq. (53) tend to zero ($\kappa_1 \to 0, \kappa_2 \to 0$) we get the situation of "interaction" of two linear waves ($\phi_n^{(bb)} \to \phi_n^{(\mathrm{LL})}$).

$$\phi_n^{(\mathrm{LL})} \approx \zeta_1\cos\vartheta_1 + \zeta_2\cos\vartheta_2, \tag{85}$$

where $\zeta_1, \zeta_2 \ll 1$ are the small amplitudes of the linear waves.

$$\vartheta_j = k_j n - \omega_{\mathrm{L}j} t, \quad \omega_{\mathrm{L}j} = 2\sin\frac{k_j}{2}, \quad (j = 1, 2). \tag{86}$$

Expressions (63)–(66) are reduced in the following way:

$$\Delta X_1 = 0 = \Delta X_2, \tag{87}$$

$$\Delta\Phi_1 = 0, \tag{88}$$

$$\Delta\Phi_2 = \pm 2\pi. \tag{89}$$

Obviously, during the "collision" of two linear waves there are no shifts, except the shift of the one of the linear waves' phase of oscillations by the value 2π. In the linear case the linear superposition principle is valid.

The results obtained can be used in the experimental investigations of the high accuracy data storage and transmission in the electrical solitonic transmission lines.[6] The phase-shift analysis is of vital importance in establishing the continuum density of states. This density of states will be, in turn, important for the calculation of the classical statistical mechanics of the Hirota lattice model.

6. Conclusion

For the nonlinear lumped self-dual network equations and equivalent Hirota lattice equation the pair interaction processes of the breathers with each other as well as with one-parametric solitons (kinks, antikinks) and linear waves are investigated. Using the obtained explicit expressions for the "kink-breather" and "breather-breather" solutions we completely analyse the collision process of the one-parametric soliton and breather and a scattering of two moving breathers. To get the case "kink-breather" one has to "bound" two solitons in breather in three-soliton solution by choosing the complex conjugative parameters. And in order to get "breather-breather" solution one has to "bound" one-parametric solitons in breathers (solitons with numbers 1 and 3 we have "bound" in one breather, while the solitons with numbers 2 and 4 we have "bound" in another breather). The analytical expressions for the scattering data, the center-of-mass shifts of solitons and phase shifts of internal oscillations of breathers have been obtained. The center-of-mass shifts correspond to the attraction of one-parametric soliton (kink or antikink) and breather and to the attraction of either two solitons or two breathers with each other. It means that one-parametric solitons and breathers of the nonlinear lumped self-dual network equations and Hirota lattice equation interact with each other by means of the effective short-range forces of attraction.

The special cases of collision processes have been discussed: the interaction of kink and standing breather; the interaction of kink and linear wave; the interaction of moving and standing discrete breather, the interaction of discrete breather and linear wave, the linear superposition of two linear waves.

Results obtained can be used for the quantitative description of the discrete one-parametric soliton and breather propagation and reflection from the free and fixed boundaries in the lattice and in constructing the low-temperature thermodynamics.

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