Collège Doctoral
Lille Nord de France

VISITING PROFESSORS 2018
Course proposal with the inviting referent Professor (must be filled)

1. Motivations of the invited Professor (500 prints)

Integrable nonlinear PDE are not only a class of nonlinear equations allowing detailed study through "linearization" by Inverse Scattering Transform (IST) method, but also provide a guide for studying non-integrable equations (i.e., in studying solitary wave solutions). The Riemann-Hilbert form of the IST is particularly well adopted for studying various asymptotic regimes. Currently, the most experts in this field are from the USA and the former Soviet countries. I believe that the French mathematical community will benefit from the popularization of this approach.

2. Description of the proposal in terms of PhD training including experience of the Visiting Professor (8000 prints)

Riemann-Hilbert (RH) problems are boundary-value problems for sectionally analytic functions in the complex plane. It is a remarkable fact that a vast array of problems in mathematics, mathematical physics, and applied mathematics can be posed as Riemann-Hilbert problems. These include radiation, elasticity, hydrodynamic, diffraction problems, orthogonal polynomials and random matrix theory, nonlinear ordinary and partial differential equations. In applications, the data for a RH problem depend on external parameters, which are physical variables (space, time, matrix size, etc), and, in turn, the solution depends on these parameters as well. It is this dependence that we are interested in, when speaking about the RH problem as a method for studying problems from one or another domain.

The representation of a solution to a *nonlinear* partial differential equation (PDE) in terms of a solution of the associated RH problem can be viewed as a nonlinear analogue of the contour integral representation for a linear PDE. It provides means to efficiently study not only the existence and uniqueness problems for a class of nonlinear PDE, but (i) to derive detailed asymptotics of solutions of initial value problems and initial boundary value problems for such equations and (ii) to accurately evaluate the solutions inside as well as outside asymptotic regimes. Recent literature on applications of the RH problem includes the monographs [1-3].

The main aim of the proposed course "**Riemann-Hilbert problems and integrable nonlinear partial differential equations**" is to introduce the Inverse Scattering Transform method, in the form of the RH problem, for studying *integrable nonlinear differential equations* and to illustrate the fruitfulness of the method by studying the long-time asymptotics of solutions of such equations. As a prototype model, we will use the nonlinear Schrödinger equation, which is a basic models of nonlinear wave propagation (for instance, in the fiber optics).

The course consists of the introductory part and three main parts.

In the Introductory part (2 hours), we (i) introduce elements of the theory of Cauchy integrals (limiting values, Sokhotski-Plemelj formulas) and its applications to the solution of the scalar additive and scalar multiplicative RH problems; (ii) discuss the matrix multiplicative RH problem using the theory of singular integral operators (solvability, uniqueness, problem index).

Part I (4 hours) contains the construction of the Riemann-Hilbert representation of a solution of the Cauchy problem in the case of decaying initial data. It involves the following topics:

1. Lax pair representation of a nonlinear PDE;
2. Jost solutions of the Lax pair equations associated with the nonlinear Schrodinger equation (NSE);
3. Scattering relation between the Jost solutions; spectral functions; continuous and discrete spectrum;
4. Integral equations for the Jost solutions; analyticity and boundedness properties of the Jost solutions and spectral functions;
5. Construction of the RH problem as an interpretation of the scattering relation;
6. Residue conditions at points of the discrete spectrum;
7. Reducing the meromorphic RH problem to a holomorphic one.
8. Solvability and uniqueness of the RH problem;
9. Representation of the solution of the NLS equation in terms of the solution of the RH problem.
10. Soliton solutions by the RH problem approach.

Part II (4 hours) contains the long-time asymptotic analysis of the solution of the NLS equation via the asymptotic analysis of the associated RH problem.

1. Recall of the classical (linear) steepest descent method for evaluating contour integrals and its application to the solutions of the linearized NLS;
2. Basic ideas of the *nonlinear* steepest descent method: analysis of the "signature table"; triangular (algebraic) factorizations of the jump matrix of the RH problem; deformation of the contour in the RH problem; analytic approximations;
3. Rescaling near the stationary points;
4. Explicit solution of the model RH problem associated with the stationary point; parabolic cylinder functions;
5. Derivation of the main asymptotic term;
6. Error estimates using the singular integral equations associated with the RH problem.

Part III (2 hours) contains an introduction to the RH problem approach in the case of non-decaying solutions:

1. Analysis of the boundedness properties of the jump matrix of the RH problem;
2. Modification of the phase function using the "g-function mechanism";
3. Reduction, in the long-time limit, to a model RH problem with piece-wise constant (with respect to the spectral parameter) jump matrix;
4. Solution of the model problem in terms of Riemann theta-functions and Abel integrals on a Riemann surface.

I have been teaching a one-semester version of the proposed course for 3 last years at V.Karazin Kharkiv National University (Ukraine). Besides, I teach a graduate course on inverse spectral and inverse scattering problems for linear differential operators, which is directly related to the proposed course since inverse problems for linear equations from the Lax pairs associated with integrable nonlinear equations constitute a basic ingredient of the Inverse Scattering Transform method for integrating nonlinear equations.

In 2000-2002, in the framework of program « PAST-ECO triennal », I taught the master courses (DEA de Mathématiques) at University Paris-7 « Problème de Riemann Hilbert et applications » and «Introduction à la théorie des problèmes inverses », and I directed 5 trainings (stages du DEA de Mathématiques) on inverse problems and their applications.

References

1. P.Deift, Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach, AMS, Providence, Rhode Island, 2000.
2. A.S. Fokas, A.R.Its, A.A.Kapaev and V.Yu.Novokshenov, Painleve Transcendens : The Riemann-Hilbert Approach, Mathematical surveys and monographs 128, AMS, 2006.
3. T.Trogdon and S.Olver, Riemann–Hilbert Problems, Their Numerical Solution, and the Computation of Nonlinear Special Functions, SIAM, Philadelphia, 2016.

3. Research collaboration project, if necessary (2000 prints)

There is a class of nonlinear integrable equations that can be viewed as short wave limits of other integrable, peakon-type (Camassa-Holm-type) equations. Together with their "full" counterparts, they have a particular physical meaning and turn to be formally integrable since they also have a Lax pair representation. Particular examples are the Hunter-Saxton equation, the Vakhnenko-Ostrovsky equation, and the short pulse equation, which are the short wave limits of respectively the Camassa-Holm equation, the Degasperis-Procesi, and the modified Camassa-Holm equation. A common feature of these equations is that the spatial operator from the respective Lax pair is more simple than that for the full equation, which suggests that the analysis of particular problems for these equations, like initial boundary value problems and the analysis of the long time behaviour of solutions, could be made in a more complete way. Another important feature is that the mechanism of wave breaking for these equations is more transparent: it is presumably related to the change of coordinates (from the Lagrangian to the original ones), while the solution in the Lagrangian coordinates may exist globally in time.

Since 2011 I have been collaborating with Prof. L.Zielinski (LMPA, Université du Littoral Côte d'Opale) in the field of integrable nonlinear equations (4 published papers). The proposed project aims at further developing this collaboration, focusing on the inverse scattering approach for peakon-type equations and their short-wave limits, in view of analyzing the long time behavior of solutions of initial and initial boundary value problems and providing efficient sufficient conditions of wave breaking. A major difficulty in the implementation of the method to *initial boundary value problems* is the fact that it requires the characterization of the so-called generalized Dirichlet-to-Neumann map, which characterizes the unknown boundary values in terms of the given ones. Particular physically significant cases of boundary conditions are those oscillating with time (asymptotically periodic or quasi-periodic). The proposed project aims at developing the method in the direction of constructing general sets of potentially asymptotically admissible boundary values (necessary conditions) for the peakon-type equations.

4. Audience (in consultation with the Doctoral School)

Etudiants de Master Mathématiques Recherche de l'école doctorale SPI spécialité Mathématiques, chercheurs en mathématiques et physique mathématiques, notamment, en équations aux dérivées partielles